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Shore-Slaman Join Theorem for continuous degrees and its applications in Descriptive Set Theory

Tesi di Laurea

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"Never attribute to malice that which is adequately explained by stupidity."

– Hanlon's razor

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Abstract

This thesis explores the structure of continuous degrees, an extension of Turing degrees in the context of separable metrizable spaces. Using tools from Effective Descriptive Set Theory and Computability, we investigate some of their properties and applications to Borel functions between Polish spaces. The techniques employed rely on deep results from both subjects, with a central role played by the Shore-Slaman Join Theorem.

Effective Descriptive Set Theory is traditionally developed in the context of separable metric spaces, we present here an alternative approach due to Alain Louveau in the broader setting of second countable topological spaces. This allows to extend the Shore-Slaman Join Theorem in the context of separable metrizable spaces using the techniques presented by Vassilios Gregoriades, Takayuki Kihara, and Keng Meng Ng in the paper *Turing degrees in Polish spaces and decomposability of Borel functions.* As a by-product, we present a characterization of Σ^0_1-recursive functions between these "effective" second countable topological spaces in the framework of Computable Analysis, that is analogous to the classical characterization of continuous functions.

We present two applications of the Shore-Slaman Join Theorem. The first is a decomposability result for Borel functions, previously established by Vassilios Gregoriades, Takayuki Kihara, and Keng Meng Ng in the paper already cited. The second is an original work that builds on an argument of Patrick Lutz presented in the paper *The Solecki Dichotomy and the Posner-Robinson theorem are almost equivalent*, and provides a weak version of the Solecki Dichotomy (with weak continuous reducibility instead of topological embeddability) for functions from Polish spaces to separable metrizable spaces. Moreover, under the Axiom of Determinacy, the same argument can be extended to functions between separable metrizable spaces.

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The first person I would like to thank is my supervisor, Raphaël Carroy. His guidance and way of thinking have incredibly improved my understanding of the objects in (Effective) Descriptive Set Theory and Computability. Moreover, his patience and availability allowed me to explore more topics than the ones we treated in this thesis, and to also learn a method for proceeding while researching and teaching, rather than just notions.

An important role in this work was played by Gabriel Debs[1], who (almost) without knowing me agreed to work with me during my time in Paris last year and identified [GKN21] as starting point for this thesis. I am very grateful for his availability and his initial support.

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[1] Actually, I should also thank Dominique Lecomte who first introduced me to him.

input. So... feel guilty about it! (I guess?)

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Orazio

[2] Do not hate me for this.

Introduction

The objective of this thesis is to study applications of results from Computability Theory, about Turing degrees, to (Effective) Descriptive Set Theory. The starting point of this work was thus the article [GKN21] where the authors develop the theory of continuous degrees in the context of separable metric spaces (previously introduced by [Mil04]). They generalize the Shore-Slaman Join Theorem and apply it to get a decomposability result for Borel functions.
However, we also have two other purposes. On the one hand, we define the objects of Effective Descriptive Set Theory in a more general framework than the usual one (as for example [Mos09]). We follow, indeed, the approach presented in [Lou19] that allows to develop effectivity for second countable spaces. On the other hand, we recast the notion of admissible representation on second countable T_0 spaces, which is the starting point of Computable Analysis, in the effective framework of [Lou19]. In this way we make explicit a characterization of Σ^0_1-recursive functions that was implicitly used in [GKN21].

To the best of my knowledge, most of the materials from [Lou19] were never published (at least in the form presented in this work) and were originally intended for a course. In this thesis, I present only a small selection of the results from these notes, following the pdf file created by Yann Pequignot, Raphaël Carroy, and Kevin Fournier, rather than Louveau's original handwritten notes. Furthermore, I slightly modified some definitions and theorems to adapt them to my goals. Hence, I take responsibility for any possible errors or misunderstandings, and the reader should attribute them to my stupidity or sloppiness. I give full credit to the original author for the innovations and the insights in his handwritten notes. Finally, I would like to take this opportunity to express my gratitude to Alain Louveau and the authors of the pdf file. Without this material, this thesis would have been much less interesting and less complete.

We require the reader to have some basic knowledge about Computability, for example, the material found in [Coo04] or in [MP22] is more than sufficient. In particular, we suppose that the reader is familiar with the concept of partial computable function on the natural numbers (and with its relativization to oracles). It could be useful to have some knowledge also in classical Descriptive Set Theory, for us the main reference is [Kec95].

Structure of the thesis

- **Chapter 1:** In this chapter, we develop Effective Descriptive Set Theory following unpublished notes of Alain Louveau [Lou19] in the framework of basic spaces, that are an effective counterpart of second countable spaces (Section 1.1.1). Then we define an effective counterpart for separable metrizable spaces: the recursive spaces (Section 1.1.2). They are the main background objects for the thesis. In Section 1.2 we introduce boldface and lightface pointclasses, with a focus on parametrization systems and Γ-recursive functions. Finally, in Section 1.3 we briefly present the relativization of effective recursive spaces that allows to work on separable metrizable spaces with the techniques developed before.
- **Chapter 2:** In Section 2.1.2 we first introduce the approach of Computable Analysis to characterize partial continuous functions in second countable T_0 spaces and then show how to restate a similar characterization for partial Σ^0_1-recursive functions between T_0 basic spaces. Then in Section 2.1.3, we present some representations that are used in Section 2.2.2. In Section 2.2 we first introduce the structure of continuous degrees and an extension of the Turing jump operator that preserves such degree structure following the exposition in [GKN21] (Section 2.2.1). Finally, we present a proof of the Shore-Slaman Join Theorem (Section 2.2.2).
- **Chapter 3:** In this chapter we show how to apply the Shore-Slaman Join Theorem to prove results on the decomposability of Borel functions as proved in [GKN21] (in Section 3.1). Finally, in Section 3.2, we briefly introduce the Solecki Dichotomy (Section 3.2.1) and explain why it implies its weak version proved in [Lut23] (Section 3.2.2). Then, in Section 3.2.3 we present our result in the context of Borel functions between recursive spaces and explain how to extend it to functions defined on Polish spaces.

Notation

We recall some notation that is used in the thesis. We denote with ω the set of natural numbers and with ω^ω the Baire space. Given s, t finite strings in $\omega^{<\omega}$ we denote with $s^\frown t$ their concatenation. With a slight abuse of notation, we use the same symbol for concatenate a finite string $s \in \omega^{<\omega}$ with an infinite string $x \in \omega^\omega$: $s^\frown x$. Moreover, when the concatenated string is made of only one character (say $n \in \omega$ for example) we write $n^\frown s$ instead of $(n)^\frown s$. Given two infinite strings $x, y \in \omega^\omega$, the join $x \oplus y$ is defined by:

$$(x \oplus y)(n) = \begin{cases} x(m) & \text{if } n = 2m \\ y(m) & \text{if } n = 2m+1 \end{cases}$$

Similarly, given two sets of natural numbers $A, B \subseteq \omega$ we define the join set $A \oplus B$ as $\{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$. We fix throughout the thesis an effective enumeration of the finite strings in $\omega^{<\omega}$ that we denote $\boldsymbol{s} = (s_i)_{i \in \omega}$ defined as follows:

$$\begin{aligned} \boldsymbol{s} : \omega &\to \omega^{<\omega} \\ n &\mapsto \boldsymbol{s}(n) = s_n = (((n)_1)_0, \dots, ((n)_1)_{(n)_0 - 1}) \end{aligned}$$

where n is considered first as a pair $((n)_0, (n)_1)$ and then $(n)_1$ is considered as $(n)_0$-tuple (using the effective coding for strings of fixed length). We consider such a coding because it is bijective (and recursive in both direction) and, moreover, it makes the following relations recursive (in the codes): $s \prec t$, $s \preceq t$, $s = t$, $\ell(s) = k$ and $i < \ell(s) \wedge s(i) = k$. Where $\ell(s)$ is the length of the string s and $s \prec t$ means that t properly extends s, that is $\exists n \in \omega (t \restriction n = s \wedge \ell(t) > \ell(s))$.
Accordingly to the notation used for the Arithmetical Hierarchy (see [Coo04, Section 10.5]) we denote by $\Delta^0_1(\omega)$ the set of recursive relations on ω and by $\Sigma^0_1(\omega)$ the set of semirecursive relations (and their relativized-to-α version are $\Delta^{0,\alpha}_1(\omega)$ and $\Sigma^{0,\alpha}_1(\omega)$). We will extend this hierarchy to recursive spaces while maintaining the same notation. We denote partial functions by $f : X \rightharpoonup Y$. In particular, given a partial function f, we write $f(x)\downarrow$ if f is defined on x (i.e. if $x \in \text{dom}(f)$) and $f(x)\uparrow$ otherwise. The e-th partial recursive function on the natural numbers ω is denoted by φ_e, and their domains, that give an enumeration of all the semirecursive sets on ω, are denoted by W_e. We use a similar convention for partial α-recursive functions with domain in the product space ω^k: in this case, the functions are denoted by $\varphi^{k,\alpha}_e$ and the domains by $W^{k,\alpha}_e$.

Chapter 1

Recursivity on Polish spaces, lightface and boldface hierarchies

This chapter covers the basics of the Effective Descriptive Set Theory (with a focus on the lightface pointclasses and Γ-recursive functions). We follow unpublished notes of Alain Louveau [Lou19] and expand them with some results from Moschovakis [Mos09]. The main difference between the Moschovakis approach and the one of Louveau is that the first one works with effectivization of (complete) separable metric spaces (called recursively presented metric spaces) while Louveau's approach involves an effectivization of the topology and allows us to work with separable metrizable spaces. Some definitions and results from [Lou19] are unpublished and originally intended as material for a course. We recast all the results of the paper [GKN21] and ours in this framework, because we believe it to be more natural when working with topological spaces.

1.1 Recursive spaces and recursively presented metric spaces

Descriptive Set Theory mainly deals with separable metrizable spaces[1] and Polish spaces, here we present effective counterparts to these concepts. Our presentation is in the spirit of [Lou19], and, unless explicitly mentioned otherwise, all notions and results come from that source, but with a different organization.

[1]Recall that for metrizable spaces being separable is equivalent to being second countable.

1.1.1 Basic spaces

Definition 1.1. A **basic space** $\mathcal{X}$ is a pair $(X, (V_n)_{n\in\omega})$ where X is a second countable topological space, $(V_n)_{n\in\omega}$ is an enumeration (possibly with repetitions) of a countable basis of the topology of X[2], and moreover there is a Σ^0_1 relation $R \in \omega^3$ such that:

$$x \in V_m \cap V_n \Leftrightarrow \exists p \in \omega(x \in V_p \wedge R(m,n,p))$$

As observed in [Val21], the condition for being a basic space can be stated for finite intersection requiring that there is $R^* \in \Sigma^0_1(\omega^{<\omega} \times \omega)$ such that:

$$x \in \bigcap_{i<\ell(\sigma)} V^{\mathcal{X}}_{\sigma(i)} \Leftrightarrow \exists p \in \omega(x \in V^{\mathcal{X}}_p \wedge R^*(\sigma, p))$$

Although it seems stronger, it is equivalent. Indeed, it is sufficient to consider as relation $R^*(\sigma,p) = \bigwedge_{i,j<\ell(\sigma)} R(\sigma(i),\sigma(j),p)$. Therefore, we use interchangeably both versions depending on what is most convenient.

Remark 1.2 (A remark on the notation for basic spaces)**.** As we did in the previous definition, we denote a topological space with X (or (X,τ_X) when we want to stress the topology) while for a basic space $(X,(V_n)_{n\in\omega})$ we use the calligraphic symbol $\mathcal{X}$. This is because a basic space is a topological space with a fixed enumeration of the basis (namely the one that makes it basic) and we want to stress the difference between these two concepts.

Proposition 1.3 (Subspaces and products of basic spaces)**.**

- Given a basic space $\mathcal{X} = (X, (V^{\mathcal{X}}_n)_{n\in\omega})$ and $Y \subseteq X$, then $\mathcal{Y} = (Y, (V^{\mathcal{Y}}_n)_{n\in\omega})$ with respect to the subspace topology and $V^{\mathcal{Y}}_n = V^{\mathcal{X}}_n \cap Y$ is a basic space (as witnessed by the same relation R).
- Given basic spaces $(X_0, (V^0_n)_{n\in\omega}), \ldots, (X_{k-1}, (V^{k-1}_n)_{n\in\omega})$, then $X = \prod_{i=0}^{k-1} X_i$ endowed with the product topology is a basic space, with $V^{\mathcal{X}}_n = \prod_{i=0}^{k-1} V^i_{(n)_i}$ and the relation defined by

$$\forall m,n,q \in \omega(R(m,n,q) \Leftrightarrow \bigwedge_{i<k} R^i((n)_i,(m)_i,(q)_i)$$

- Given $((X_i, (V^i_n)_{n\in\omega}))_{i\in\omega}$ sequence of basic spaces (witnessed respectively by the relations $R^i \subseteq \omega^3$), if the relation $S \subseteq \omega^4$ defined as:

$$\forall p,m,n,q \in \omega(S(p,m,n,q) \Leftrightarrow R^p(n,m,q))$$

  is Σ^0_1, then $X = \prod_{i\in\omega} X_i$ endowed with the product topology is a basic space considering $V^{\mathcal{X}}_n = \prod_{i<\ell(s_n)} V^i_{s_n(i)} \times \prod_{i\geq\ell(s_n)} X_i$.

[2]The V_ns are not necessarily not empty.

The previous proposition gives the possibility of considering substructures. This is the main difference between basic spaces and recursively presented metric spaces (based on enumerating countable dense sets).
Moreover, the space ω is basic with the enumeration of the base given by $V_n^\omega = \{n\}$. This provides a canonical structure as basic space for each product space $X \times \omega$, with $\mathcal{X} = (X, (V_n^{\mathcal{X}})_{n\in\omega})$ a basic space.

Σ_1^0 sets

We now introduce the "effective topology" for a basic space $\mathcal{X}$.

Definition 1.4. A subset A of a basic space $\mathcal{X}$ is called $\Sigma_1^0(\mathcal{X})$ (also said **effectively open** in $\mathcal{X}$) if there is a Σ_1^0 set A^* in ω such that

$$x \in A \Leftrightarrow \exists n \in \omega (x \in V_n \wedge n \in A^*)$$

Note that the effectively open sets of the basic space ω, by the definition above, are exactly the Σ_1^0 sets as defined in computability, so our notation is not ambiguous for this space (and the same remark applies for the product spaces ω^k).

Remark 1.5. By considering as A^*, the empty set $\varnothing$, ω, and the singletons $\{n\}$, one gets immediately that for every basic space $\mathcal{X}$, the empty set, X itself, and each $V_n^{\mathcal{X}}$ in the basis are all $\Sigma_1^0(\mathcal{X})$ sets. Moreover, as for the semirecursive sets in ω, Σ_1^0 is closed under union, intersection, bounded quantification and existential quantification over ω (denoted by $\exists^0$).

Proposition 1.6 (Separation of variables)**.** Given $\mathcal{X}$, $\mathcal{Y}$ basic spaces, a subset $A \subseteq X \times Y$ is Σ_1^0 if and only if there is a $B \in \Sigma_1^0(\omega^2)$ such that:

$$(x, y) \in A \Leftrightarrow \exists p, q \in \omega (x \in V_p^{\mathcal{X}} \wedge y \in V_q^{\mathcal{Y}} \wedge B(p, q))$$

and similarly for any finite product of basic spaces.

Theorem 1.7 (Universal Σ_1^0 for open sets)**.** Given $\mathcal{X}$ basic space, there is a set $G \in \Sigma_1^0(\omega^\omega \times \mathcal{X})$ such that

$$\forall U \in \tau_X \exists \alpha \in \omega^\omega \forall x \in X (x \in U \Leftrightarrow (\alpha, x) \in G)$$

where τ_X is the topology of X.

We call a set G with this property *universal* for all open subsets of X. A formal definition of *universal set* (in the more general context of pointclasses) is given in subsection 1.2.3.

We observe that this result is stated in [Lou19, Theorem 3.3.1] for recursive spaces (that we introduce in the next section), however the same argument can be carried out in any basic space.

Proof. Notice that any open set $U \in \tau_X$ is the union of a subfamily of the basis $(V_n^{\mathcal{X}})_{n\in\omega}$, hence to any open set we can associate the function $\alpha_U : \omega \to \omega$ that enumerates the indices n such that $V_n^{\mathcal{X}} \subseteq U$. It may be that no $V_n^{\mathcal{X}}$ is empty, but we need the empty union to include also the empty set $\varnothing$, hence we define $G \in \Sigma_1^0(\omega^\omega \times \mathcal{X})$ by:

$$(\alpha, x) \in G \Leftrightarrow \alpha(0) \neq 0 \wedge \exists p \in \omega(p \geq 1 \wedge x \in V_\alpha(p)^{\mathcal{X}})$$

In this way:

- if $\alpha(0) = 0$, then $G_\alpha = \varnothing$
- for any non empty open set we simply consider $\alpha = 1^\frown\alpha_U$, and $U = G_\alpha$

□

Theorem 1.8 (Universal Σ_1^0 set for effectively open sets)**.** Given $\mathcal{X}$ basic space, there is a set $G \in \Sigma_1^0(\omega \times \mathcal{X})$ such that:

$$\forall U \in \Sigma_1^0(\mathcal{X}) \exists e \in \omega \forall x \in X(x \in U \Leftrightarrow (e, x) \in G)$$

Also in this case we call it *universal* for all $\Sigma_1^0(\mathcal{X})$.

Proof. The halting set $H = \{(i, j) \mid \varphi_i(j) \downarrow\}$ is universal for all $\Sigma_1^0(\omega)$. We define then:

$$G(e, x) \Leftrightarrow \exists p \in \omega(x \in V_p^{\mathcal{X}} \wedge H(e, p))$$

Clearly $G \in \Sigma_1^0(\omega \times \mathcal{X})$, and by the universality of H and the definition of effectively open it is universal for $\Sigma_1^0(\mathcal{X})$. □

Σ_1^0-recursive functions

Definition 1.9. Given $\mathcal{X}$, $\mathcal{Y}$ basic spaces, a function $f : \mathcal{X} \to \mathcal{Y}$ is **Σ_1^0-recursive** if its diagram is Σ_1^0, that is:

$$D_f = \{(x, n) \mid f(x) \in V_n^{\mathcal{Y}}\} \in \Sigma_1^0(\mathcal{X} \times \omega)$$

In particular, if $f : \mathcal{X} \to \mathcal{Y}$ is Σ_1^0-recursive, then there exists a $B \in \Sigma_1^0(\omega^2)$:

$$f(x) \in V_n^{\mathcal{Y}} \Leftrightarrow \exists p \in \omega(x \in V_p^{\mathcal{X}} \wedge B(p, n))$$

We now recall some basic facts about Σ_1^0-recursive functions that we use in the thesis.

Proposition 1.10 (Closure under recursive substitution)**.** Given $\mathcal{X}$, $\mathcal{Y}$ basic spaces, $f : \mathcal{X} \to \mathcal{Y}$ Σ^0_1-recursive and $A \in \Sigma^0_1(\mathcal{Y})$, then $f^{-1}[A] \in \Sigma^0_1(\mathcal{X})$.

Proof. Given $A \in \Sigma^0_1(\mathcal{Y})$ there is an $A^* \in \Sigma^0_1(\omega)$ such that

$$y \in A \Leftrightarrow \exists n \in \omega(y \in V^{\mathcal{Y}}_n \wedge n \in A^*)$$

therefore, we get

$$x \in f^{-1}[A] \Leftrightarrow \exists p \in \omega(f(x) \in V^{\mathcal{Y}}_p \wedge p \in A^*) \Leftrightarrow \exists p \in \omega(D_f(x,p) \wedge p \in A^*)$$

and the last term is $\Sigma^0_1(\mathcal{X})$ because Σ^0_1 is closed under finite conjunctions. □

Proposition 1.11 (Recursive function in product spaces)**.** Given $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{X}_0, \dots, \mathcal{X}_k$ basic spaces:

1. $f : \mathcal{X} \to \prod_{i=0}^k \mathcal{X}_i$ is Σ^0_1-recursive if and only if its projections $\pi_i \circ f : \mathcal{X} \to \mathcal{X}_i$ are Σ^0_1-recursive
2. $f : \mathcal{X} \to \mathcal{Y}^\omega$ is Σ^0_1-recursive if and only if the function

$$\begin{aligned} g : \mathcal{X} \times \omega &\to \mathcal{Y} \\ (x,n) &\to g(x,n) = \pi_n \circ f(x) \end{aligned}$$

   is recursive (recall that π_n is the projection on the n-th factor).

Proof. The proof of both point is quite immediate, in particular:

1. It suffices to observe that the following equivalences hold

$$\pi_i \circ f(x) \in V^{\mathcal{X}_i}_n \Leftrightarrow \exists n_0, \dots, n_{i-1}, n_{i+1}, \dots, n_k \in \omega \left(f(x) \in V^{\prod \mathcal{X}_i}_{\langle n_0, \dots, n_{i-1}, n_{i+1}, \dots, n_k\rangle}\right)$$

   where $\langle n_0, \dots, n_{i-1}, n_{i+1}, \dots, n_k\rangle \in \omega$ is the code corresponding to the finite string $(n_0, \dots, n_{i-1}, n_{i+1}, \dots, n_k)$ and

$$f(x) \in V^{\prod \mathcal{X}_i}_n \Leftrightarrow \pi_0 \circ f(x) \in V^{\mathcal{X}_0}_{(n)_0} \wedge \dots \wedge \pi_k \circ f(x) \in V^{\mathcal{X}_k}_{(n)_k}$$

2. It suffices to observe that

$$g(x,i) = \pi_i \circ f(x) \in V^{\mathcal{Y}}_n \Leftrightarrow \exists k \in \omega \left(f(x) \in V^{\mathcal{Y}^\omega}_k \wedge \ell(s_k) > i \wedge s_k(i) = n\right)$$

   and

$$f(x) \in V^{\mathcal{Y}^\omega}_n \Leftrightarrow \forall i < \ell(s_n) \left(g(x,i) \in V^{\mathcal{Y}}_{S_n(k)}\right)$$

□

Definition 1.12. Given $\mathcal{X}$, $\mathcal{Y}$ basic spaces, $f : \mathcal{X} \to \mathcal{Y}$ is a **recursive isomorphism** if it is Σ^0_1-recursive, bijective, and has Σ^0_1-recursive inverse.

Recursively presented metric spaces are basic

Effective Descriptive Set Theory is often developed in the context of recursively presented metric spaces (see for example [Mos09, Chapter 3]). The more general approach of Louveau is coherent with such context, indeed for any recursively presented metric space we can find a basis that makes is basic (and we'll see that also other concepts introduced by Louveau fit them). We fix once and for all $(q_i)_{i\in\omega}$ an effective enumeration of $\mathbb{Q}^+$.

Definition 1.13. Let (X, d) be a separable metric space and $\boldsymbol{r} = (r_i)_{i\in\omega}$ an enumeration (possibly with repetitions) of a dense subset of X. We say that $\boldsymbol{r}$ is a **recursive presentation** of X if the relations on ω^3

$$P(i,j,k) \Leftrightarrow \mathrm{d}(r_i, r_j) \leq q_k$$
$$Q(i,j,k) \Leftrightarrow \mathrm{d}(r_i, r_j) < q_k$$

are recursive. The structure $(X, \mathrm{d}, \boldsymbol{r})$ is called **recursively presented metric space**. If moreover (X, d) is complete, then $(X, \mathrm{d}, \boldsymbol{r})$ is called **recursively presented Polish space**.

As observed in [Lou19], we remark that it would be more adequate to call the recursively presented Polish spaces as *recursively presented complete metric spaces*, as the complete distance is an explicit part of the structure.[3]

Example 1.14 (Examples of recursively presented Polish spaces)**.** We present here some useful examples of recursively presented Polish spaces.

- On ω we consider the *discrete metric*, that is:

$$\mathrm{d}(n,m) = \begin{cases} 1 & \text{if } n \neq m \\ 0 & \text{if } n = m \end{cases}$$

  as dense sequence we consider $\boldsymbol{r} = (i)_{i\in\omega}$. Hence:

$$P(i,j,k) \Leftrightarrow \mathrm{d}(r_i, r_j) \leq q_k \Leftrightarrow q_k \geq 1 \vee i = j$$
$$Q(i,j,k) \Leftrightarrow \mathrm{d}(r_i, r_j) < q_k \Leftrightarrow q_k > 1 \vee i = j$$

  are clearly recursive.

[3] Actually, the definition given in [Lou19] for *recursively presented metric spaces* is more general as it consider, instead of the predicates P and Q (in $\Delta^0_1(\omega^3)$), a predicate (on ω^4):

$$A(i,j,k,l) \Leftrightarrow q_l < \mathrm{d}(r_i, r_j) < q_k$$

and require it to be semirecursive. However, this doesn't affect the results presented in this thesis.

- On ω^ω (and similarly on 2^ω) we consider the *prefix metric*, that is:

$$\mathrm{d}(\alpha,\beta) = \begin{cases} \frac{1}{2^{n+1}} & \text{if } \alpha \neq \beta \wedge n = \min\{m \in \omega \mid \alpha(m) \neq \beta(m)\} \\ 0 & \text{if } n = m \end{cases}$$

as dense sequence we consider an effective enumeration of all eventually-zero sequences $\boldsymbol{r} = (s_i{}^\frown 0^\infty)_{i\in\omega}$ where $(s_i)_{i\in\omega}$ is the fixed enumeration of $\omega^{<\omega}$. Then:

$$P(i,j,k) \Leftrightarrow \mathrm{d}(r_i,r_j) \leq q_k \Leftrightarrow \exists n \in \omega(r_i(n) \neq r_j(n) \wedge 2^{-n-1} \leq q_k) \vee \\ \forall n \in \omega(q_k \leq 2^{-n-1} \Rightarrow r_i(n) = r_j(n))$$

is recursive and similarly is Q.

- On $[0,1]$ we consider the usual distance $\mathrm{d}(x,y) = |x-y|$ and as dense sequence $\boldsymbol{r} = q^1 = (q^1_n)_{n\in\omega}$ an effective enumeration of $\mathbb{Q}^+ \cap [0,1]$. Then:

$$P(i,j,k) \Leftrightarrow \mathrm{d}(r_i,r_j) \leq q_k \Leftrightarrow |q^1_i - q^1_j| \leq q_k$$

is clearly recursive and similarly Q.

- On $[0,1]^\omega$ we consider the metric

$$\mathrm{d}(\alpha,\beta) = \sum_{n\in\omega} 2^{-n} \cdot |\alpha(n) - \beta(n)|$$

and as dense sequence we consider $\boldsymbol{r}$ with $\boldsymbol{r}(n) = (r^n_i)_{i\in\omega}$ such that

$$r^n_i = \begin{cases} q^1_{s_n(i)} & \text{if } i < \ell(s_n) \\ q^1_0 & \text{if } i \geq \ell(s_n) \end{cases}$$

where q^1 is the effective enumeration of $\mathbb{Q}^+ \cap [0,1]$. The proof that $([0,1]^\omega, \mathrm{d}, \boldsymbol{r})$ is a recursively presented Polish space because:

$$\mathrm{d}(\boldsymbol{r}_i,\boldsymbol{r}_j) = \sum_{n<\min\{\ell(s_i),\ell(s_j)\}} 2^{-n} \cdot |q^1_{s_i(n)} - q^1_{s_j(n)}| + \\ \sum_{\min\{\ell(s_i),\ell(s_j)\}\leq n<\ell(s_i)} 2^{-n} \cdot |q^1_{s_i(n)} - q^1_0| + \\ \sum_{\min\{\ell(s_i),\ell(s_j)\}\leq n<\ell(s_j)} 2^{-n} \cdot |q^1_{s_j(n)} - q^1_0| \leq q_k$$

so P is recursive (and similarly is Q).

Given $(X, \mathrm{d}, \boldsymbol{r})$ recursively presented metric space, we can turn canonically this space into a basic space, considering as elements of the basis:

$$V_n = B(\boldsymbol{r}((n)_0), q_{(n)_1}) = \{x \in X \mid \mathrm{d}(x, \boldsymbol{r}((n)_0)) < q_{(n)_1}\}$$

In other words, considering as countable basis for the topology of X an enumeration of all balls centered at some $\boldsymbol{r}(k)$ and with rational radius.
Moreover, if we define:

$$\begin{aligned} S(p,n) &\Leftrightarrow \mathrm{d}(\boldsymbol{r}((p)_0)), \boldsymbol{r}((n)_0))) + q_{(p)_1} < q_{(n)_1} \\ R(m,n,p) &\Leftrightarrow S(p,n) \wedge S(p,m) \end{aligned}$$

the relation $R \in \Sigma^0_1(\omega^3)$ witnesses that (X, V_n) is basic.

Proof. We have to prove that $\forall x \in X$ and $\forall m,n \in \omega$

$$x \in V_n \cap V_m \Leftrightarrow \exists p(R(m,n,p) \wedge x \in V_p)$$

$\Rightarrow$ Given $x \in V_n \cap V_m$ we have:

$$\mathrm{d}(x, \boldsymbol{r}((n)_0)) < q_{(n)_1} \wedge \mathrm{d}(x, \boldsymbol{r}((m)_0)) < q_{(m)_1}$$

we can consider a sufficiently small $l \in \mathbb{Q}^+$ such that $B(x, 2l) \subseteq V_n \cap V_m$, then by density of $\boldsymbol{r}$ there exists $i \in \omega$ such that $r_i \in B(x,l)$ hence we have that $p = \langle i,j\rangle$ (where $q_j = l$) satisfies $S(p,n)$ and $S(p,m)$, indeed:

$$\begin{aligned} \mathrm{d}(\boldsymbol{r}((n)_0), \boldsymbol{r}((p)_0)) + q_{(p)_1} &= \mathrm{d}(\boldsymbol{r}((n)_0), \boldsymbol{r}((p)_0)) + l \\ &\leq \mathrm{d}(\boldsymbol{r}((n)_0), x) + \mathrm{d}(x, \boldsymbol{r}((p)_0)) + l \\ &< \mathrm{d}(\boldsymbol{r}((n)_0), x) + 2l < q_{(n)_1} \end{aligned}$$

where the last inequality is because $B(x, 2l) \subseteq V_n$.

$\Leftarrow$ For the other direction, suppose that given $m,n \in \omega$ there is a $p \in \omega$ $x \in V_p$ with $R(m,n,p)$. Then $x \in V_n \cap V_m$ because

$$\begin{aligned} \mathrm{d}(x, \boldsymbol{r}((n)_0)) &\leq \mathrm{d}(x, \boldsymbol{r}((p)_0)) + \mathrm{d}(\boldsymbol{r}((n)_0), \boldsymbol{r}((p)_0)) \\ &< q_{(p)_1} + \mathrm{d}(\boldsymbol{r}((n)_0), \boldsymbol{r}((p)_0)) < q_{(n)_1} \end{aligned}$$

where the last inequality holds because $S(p,n)$. ☐

Remark 1.15. We should check, for each space (and especially for ω, 2^ω, ω^ω, $[0,1]$ and $[0,1]^\omega$), that the effective topology induced by considering them as recursively presented metric spaces (with the recursive presentations considered) "is the same" as their effective topology obtained considering them as basic spaces. More precisely, one can prove that the identity function considered on the same space with the two different effective bases is a recursive isomorphism.
This is relatively easy but tedious to prove and left to the reader. For this reason, we do not distinguish between the two bases, and denote them ambiguously with the calligraphic symbol as basic spaces $\mathcal{X}$. Moreover, for the spaces ω, 2^ω, ω^ω, $[0,1]$ and $[0,1]^\omega$, we don't introduce a calligraphic symbol because we consider them with only w.r.t. the basis that derives from the recursive presentations considered above.

Partial Σ^0_1-recursive functions

The framework of computable functions is usually developed for partial functions. Since we are interested in comparing concepts from Effective Descriptive Set Theory with those from Computable Analysis, we have to extend the concept of Σ^0_1-recursivity to partial functions. In this section, we extend the concepts introduced in [Mos09, Section 7A] in the more general setting of basic spaces.

Definition 1.16 ([Mos09, Section 3G])**.** Given $\mathcal{X}$, $\mathcal{Y}$ basic spaces, a partial function $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ is **Σ^0_1-recursive on** $D \subseteq \mathrm{dom}(f)$ if there exists a set $P \in \Sigma^0_1(\mathcal{X} \times \omega)$ such that

$$\forall x \in D(f(x) \in V^{\mathcal{Y}}_n \Leftrightarrow P(x,n))$$

If moreover $D = \mathrm{dom}(f)$ we say that $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ is **Σ^0_1-recursive on its domain**.

Remark 1.17. We often don't specify if the function is partial or not as the arrow notation ($\to$ or $\rightharpoonup$) classifies it unambiguously.

After the previous definition, we can define the e-th partial recursive function (on its domain) $\Phi^{\mathcal{X},\mathcal{Y}}_e$ from $\mathcal{X}$ to $\mathcal{Y}$ as the largest function induced by the e-th Σ^0_1 subset of ω^2. That is:

$$\Phi^{\mathcal{X},\mathcal{Y}}_e(x)\downarrow \Leftrightarrow \exists! y \in \mathcal{Y} \text{ such that } \forall n(y \in V^{\mathcal{Y}}_n \Leftrightarrow \exists p \in \omega(x \in V^{\mathcal{X}}_p \wedge W_e(n,p)))$$

$$\Phi^{\mathcal{X},\mathcal{Y}}_e(x) = \text{ the unique } y \text{ such that } \forall n(y \in V^{\mathcal{Y}}_n \Leftrightarrow \exists p \in \omega(x \in V^{\mathcal{X}}_p \wedge W_e(n,p)))$$

For simplicity, we introduce the following notation to indicate basic spaces that are finite cartesian products of ω and/or ω^ω.

Notation 1.18. We say that a basic space is of

- **type 0** if it is of the form ω^k for some $k \in \omega$
- **type 1** if it is of the form $\omega^k \times (\omega^\omega)^l$ for some $k, l \in \omega$.

Theorem 1.19 ([Mos09, Theorem 7A.1])**.**

1. For all basic spaces $\mathcal{X}$ and $\mathcal{Y}$, the partial function $\Phi^{\mathcal{X},\mathcal{Y}} : \omega \times \mathcal{X} \rightharpoonup \mathcal{Y}$ is Σ^0_1-recursive on its domain.
2. For all basic spaces $\mathcal{X}$ and $\mathcal{Y}$, a partial function $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ is Σ^0_1-recursive on its domain if and only if there is an $e \in \omega$ such that $f \subseteq \Phi^{\mathcal{X},\mathcal{Y}}_e$, that is:
$$f(x)\downarrow \Rightarrow f(x) = \Phi^{\mathcal{X},\mathcal{Y}}_e(x)$$

3. Given $\mathcal{X}$ space of type 0 and $\mathcal{Y}, \mathcal{Z}$ basic spaces, there is a recursive function $S^{\mathcal{X},\mathcal{Y},\mathcal{Z}} : \omega \times X \to \omega$ such that

$$\forall e \in \omega \forall x \in X \forall y \in Y (\Phi_e^{\mathcal{X}\times\mathcal{Y},\mathcal{Z}}(x, y) = \Phi_{S(e,x)}^{\mathcal{Y},\mathcal{Z}}(y))$$

Proof. 1., 2. follows from the fact that W^k is an universal set for Σ_1^0 sets in ω^k. Indeed:

$$\forall e \in \omega \forall \overline{x} \in \omega^k (W_e(\overline{x}) = H(e, \overline{x}))$$

where H is the Halting set for k-ary computable functions, that is: $H = \{(e, \overline{x}) \in \omega^{k+1} \mid \varphi_e(\overline{x}) \downarrow\}$.

3. Given $\mathcal{X} = \omega^k$, we have:

$$\begin{aligned}\Phi_e^{\mathcal{X}\times\mathcal{Y},\mathcal{Z}}(\overline{x}, y) \downarrow= z &\Leftrightarrow \forall n\Big(z \in V_n^{\mathcal{Z}} \Leftrightarrow \exists p \in \omega\big((\overline{x}, y) \in V_p^{\mathcal{X}\times\mathcal{Y}} \wedge W_e(n, p)\big)\Big)\\ &\Leftrightarrow \forall n\Big(z \in V_n^{\mathcal{Z}} \Leftrightarrow \exists p \in \omega\big(y \in V_{(p)_k}^{\mathcal{Y}} \wedge W_e(n, p)\\ &\qquad\qquad \wedge \bigwedge_{i<k}(x_i = (p)_i)\big)\Big)\\ &\Leftrightarrow \forall n\Big(z \in V_n^{\mathcal{Z}} \Leftrightarrow \exists q \in \omega \exists p \in \omega\big(y \in V_q^{\mathcal{Y}} \wedge W_e(n, p)\\ &\qquad\qquad \wedge \bigwedge_{i<k}(x_i = (p)_i) \wedge q = (p)_k\big)\Big)\end{aligned}$$

We define the function:

$$h(e, n, \overline{x}, q) = \begin{cases} 1 & \text{if } \exists p \in \omega(W_e(n, p) \wedge \bigwedge_{i<k}(x_i = (p)_i) \wedge q = (p)_k) \\ \uparrow & \text{otherwise} \end{cases}$$

This function is computable because its graph is Σ_1^0 and hence $\varphi_f = h$ for some $f \in \omega$. By the S-m-n Theorem we consider the function $S_{k+1}^2 : \omega^{k+2} \to \omega$, then:

$$\varphi_{S_{k+1}^2(f,e,\overline{x})}(n, q) = \varphi_f(e, n, \overline{x}, q) = h(e, n, \overline{x}, q)$$

Therefore:

$$\begin{aligned}\Phi_e^{\mathcal{X}\times\mathcal{Y},\mathcal{Z}}(\overline{x}, y) \downarrow= z &\Leftrightarrow \forall n(z \in V_n^{\mathcal{Z}} \Leftrightarrow \exists q \in \omega(y \in V_q^{\mathcal{Y}} \wedge W_{S_{k+1}^2(f,e,\overline{x})}(n, q)))\\ &\Leftrightarrow \Phi_{S_{k+1}^2(f,e,\overline{x})}^{\mathcal{Y},\mathcal{Z}}(y) \downarrow= z\end{aligned}$$

Since f is fixed the thesis follows. □

Theorem 1.20 (Kleene's Recursion Theorem [Mos09, Therem 7A.2])**.** Given $\mathcal{X}, \mathcal{Y}$ basic spaces and $f : \omega \times X \rightharpoonup Y$ recursive on its domain, then there exists an $\tilde{e} \in \omega$ such that

$$\forall x \in X \Big(f(\tilde{e}, x) \downarrow \Rightarrow (f(\tilde{e}, x) = \Phi_{\tilde{e}}^{\mathcal{X},\mathcal{Y}}(x))\Big)$$

Proof. We define the Σ^0_1-recursive function on its domain:

$$\begin{aligned} g : \omega \times \omega \times X &\rightharpoonup Y \\ (n, m, x) &\mapsto \Phi_m^{\omega\times\mathcal{X},\mathcal{Y}}(S(n,n,m),x) \end{aligned}$$

where $S : \omega \times \omega^2 \to \omega$ is given by Theorem 1.19 3. Therefore, by Theorem 1.19 2., there is a fixed $e \in \omega$ such that:

$$g(n,m,x)\downarrow \Rightarrow g(n,m,x) = \Phi_e^{\omega^2\times\mathcal{X},\mathcal{Y}}(n,m,x) = \Phi^{\mathcal{X},\mathcal{Y}}_{S(e,n,m)}(x)$$

Thus taking $n = e$ we have that: $\Phi_m^{\omega\times\mathcal{X},\mathcal{Y}}(S(e,e,m),x) = \Phi^{\mathcal{X},\mathcal{Y}}_{S(e,e,m)}(x)$.
Finally, if $f \subseteq \Phi_a$, taking $\tilde{e} = S(e,e,a)$ we have $\forall x \in X$ such that $f(\tilde{e},x)\downarrow$:

$$f(\tilde{e},x) = \Phi_a^{\omega\times\mathcal{X},\mathcal{Y}}(\tilde{e},x) = \Phi^{\mathcal{X},\mathcal{Y}}_{\tilde{e}}(x)$$

□

1.1.2 Recursively regular spaces and recursive spaces

We've introduced an effective counterpart to second countable topological spaces (i.e. basic spaces), we now introduce the effective counterpart to metrizable secound countable topological spaces. To this extent we introduce two concepts, namely the recursively regular spaces and the recursive spaces and, in the end, we show that they define the same class of spaces.

Definition 1.21. A basic space $\mathcal{X} = (X, (V_n)_{n\in\omega})$ is **recursively regular** if there exist two Σ^0_1 relations $S \subseteq \omega^2$ and $T \subseteq \omega^3$ such that

1. for any $j \in \omega$: $x \in V_j \Leftrightarrow \exists i(x \in V_i \wedge S(i,j))$;
2. for any $i, j \in \omega$ such that $S(i,j)$, if we consider the set

   $$P_{i,j} \coloneqq \{x \in X \mid \forall k(T(i,j,k) \Rightarrow x \notin V_k)\}$$

   then $V_i \subseteq P_{i,j} \subseteq V_j$.

Notice that, uniformly in i and j, the complement of $P_{i,j}$ is $\Sigma^0_1(\mathcal{X})$ because

$$P^c_{i,j} = \{x \in X \mid \exists k(T(i,j,k) \wedge x \in V_k)\}$$

in particular, $P_{i,j}$ is what we will call, in the next section, *effectively closed* set (or Π^0_1 set). Therefore, the conditions say, uniformly, that any set in the basis is the effective union of other sets in the basis which can be separated from it by effectively closed sets. We point that the third condition is an effective analog of the notion of regularity for topological spaces (and indeed it implies it).

Definition 1.22. A basic space $\mathcal{X}$ is said:

- **recursive** if it is recursively isomorphic to a subspace of a recursively presented metric space.
- **Polish recursive** if it is recursively isomorphic to a recursively presented Polish space.

Note that if $\mathcal{X}$ is recursive, its topology is metrizable separable.

Recursive and recursively regular spaces are the same

We recall that a topological (X, τ_X) space is T_0 (or Kolmogorov) if every pair of distinct points is topologically distinguishable. That is, if for any pair of point $x_1, x_2 \in X$ with $x_1 \neq x_2$ there exists $U \in \tau_X$ such that either $x_1 \in U \wedge x_2 \notin U$ or $x_1 \notin U \wedge x_2 \in U$.

We observe that if $\mathcal{X}$ is a T_0 basic space, in particular we get, for any pair of points $x_1, x_2 \in X$ with $x_1 \neq x_2$, an $n \in \omega$ such that either $x_1 \in V_n^{\mathcal{X}} \wedge x_2 \notin V_n^{\mathcal{X}}$ or $x_1 \notin V_n^{\mathcal{X}} \wedge x_2 \in V_n^{\mathcal{X}}$. We now prove that, for T_0 basic spaces, the two concepts of recursive space and recursively regular space are equivalent. That is:

Theorem 1.23 ([Lou19, Theorem 2.4.4])**.** A basic space $\mathcal{X}$ is recursively regular and T_0 if and only if it is recursive.

Remark 1.24. This result is stated in [Lou19] without requiring that $\mathcal{X}$ is T_0, however such assumption is necessary in the Left-to-right implication for having that the function witnessing that $\mathcal{X}$ is recursive is injective. Maybe it could be that $\mathcal{X}$ recursively regular implies that the space $\mathcal{X}$ is T_0, but we weren't able to prove or disprove it.

We follow the exposition in [Lou19] with some minor variations. Indeed, as in [Lou19] we split the results in two part (one for each direction). The right-to-left implication is quite easy and don't require any particular prerequisites, while the other implication follows as corollary of a deeper result: the Effective Urysohn Theorem. Actually the Effective Urysohn Theorem will allow us to prove that every recursively regular space is recursively isomorphic to a subspace of the recursively presented Polish space $[0,1]^\omega$. Theorem 1.23 and the effective Uryshohn Theorem 1.26 were established by Louveau but have never been published.

Proof of Theorem 1.23 - Right-to-left implication.
The fact that $\mathcal{X}$ is T_0, is clear since any recursive space is metrizable (and hence T_2). To prove the recursive regularity, we split the proof in three steps:

If $\mathcal{X}$ is a recursively presented metric space: We consider $\mathcal{X} = (X, \mathrm{d}, \boldsymbol{r})$ as a basic space by considering as enumeration of the basis the one given by

$$V_n^{\mathcal{X}} = B(\boldsymbol{r}((n)_0), q_{(n)_1}) = \{x \in X \mid \mathrm{d}(x, \boldsymbol{r}((n)_0)) < q_{(n)_1}\}$$

again we define the Σ_1^0 predicates:

$$\begin{aligned} S(m,n) &\Leftrightarrow \mathrm{d}(\boldsymbol{r}((m)_0)), \boldsymbol{r}((n)_0))) + q_{(m)_1} < q_{(n)_1} \\ R(m,n,p) &\Leftrightarrow S(p,n) \wedge S(p,m) \end{aligned}$$

recall that we've already proved that such R witnesses that $\mathcal{X}$ is basic. We define also:

$$T(m,n,k) \Leftrightarrow S(m,n) \wedge \mathrm{d}(\boldsymbol{r}((m)_0)), \boldsymbol{r}((k)_0))) > q_{(m)_1} + q_{(n)_1}$$

Thanks to the conditions on the recursively presented metric spaces we get $T \in \Sigma_1^0(\omega^3)$. We claim that S and T witness that $\mathcal{X}$ is recursively regular. Indeed:

1. given $x \in V_n^{\mathcal{X}}$, we consider $\varepsilon = q_l > 0$ small enough so that

$$\mathrm{d}(x, \boldsymbol{r}((n)_0)) < q_{(n)_1} - 2\varepsilon$$

by the density of $\boldsymbol{r}$ there is a $k \in \omega$ such that $\boldsymbol{r}(k) \in B(x, \varepsilon)$. Therefore, if we set $m = \langle k, l \rangle$ then:

$$\begin{aligned} \mathrm{d}(\boldsymbol{r}((m)_0), \boldsymbol{r}((n)_0)) &\leq \mathrm{d}(\boldsymbol{r}((m)_0), x) + \mathrm{d}(x, \boldsymbol{r}((n)_0)) \\ &< q_{(n)_1} - 2\varepsilon + q_l = q_{(n)_1} - q_l = q_{(n)_1} - q_{(m)_1} \end{aligned}$$

and hence $x \in V_m^{\mathcal{X}}$ and $S(m,n)$ holds.

2. We assume $S(m,n)$ and consider

$$P_{m,n} = \{x \in X \mid \forall k (T(m,n,k) \Rightarrow x \notin V_k^{\mathcal{X}})\}$$

we have to show that $V_m^{\mathcal{X}} \subseteq P_{m,n} \subseteq V_n^{\mathcal{X}}$.
To prove the first inclusion, we assume, towards a contradiction, that there is an $x \in V_m^{\mathcal{X}} \setminus P_{m,n}$. Then there is some $k \in \omega$ such that $x \in V_k^{\mathcal{X}}$ and $T(m,n,k)$. The first condition implies that $V_m^{\mathcal{X}} \cap V_k^{\mathcal{X}} = \varnothing$ but $T(m,n,k)$ implies that $V_m^{\mathcal{X}} \cap V_k^{\mathcal{X}} = \varnothing$, thus we have a contradiction. ↯
Now we prove the second inclusion, we observe that $S(m,n)$ not only implies that $V_m^{\mathcal{X}} \subseteq V_n^{\mathcal{X}}$, but also that the closed ball $\hat{V}_m^{\mathcal{X}} = \{x \in X \mid \mathrm{d}(x, \boldsymbol{r}((m)_0)) \leq q_{(m)_1}\}$ is contained in $V_n^{\mathcal{X}}$. Thus, to conclude it suffices to prove that $P_{m,n} \subseteq \hat{V}_m^{\mathcal{X}}$. We prove it by contraposition: given $x \notin \hat{V}_m^{\mathcal{X}}$ then it satisfies

$$\mathrm{d}(x, \boldsymbol{r}((m)_0)) > q_{(m)_1}$$

hence we consider $k \in \omega$ such that

$$\mathrm{d}(x, \boldsymbol{r}((m)_0)) > q_{(m)_1} + 2q_k$$

and an $l \in \omega$ such that $\boldsymbol{r}(l) \in B(x, q_k)$. Therefore $x \in V^{\mathcal{X}}_{\langle l,k\rangle}$ and $T(m, n, \langle l, k\rangle)$ holds and hence $x \notin P_{m,n}$.

If $\mathcal{X}$ is a subset of a recursively regular space: Given $\mathcal{Z}$ recursively regular space and $X \subseteq Z$ then we've already shown that we have a structure of basic space on it, as witnessed by the same relation R witnessing that $\mathcal{Z}$ is basic. Similarly, the same relations S and T witness that $\mathcal{X}$ is recursively regular.

If $\mathcal{X}$ is recursively isomorphic to a recursively regular space: Suppose that $\mathcal{Y}$ is recursively regular with witnesses S and T, and $\varphi : \mathcal{X} \to \mathcal{Y}$ is a recursive isomorphism. In particular, there are $A, B \in \Sigma^0_1(\omega^2)$ corresponding to the diagram of φ and its inverse, hence such that

$$\varphi(x) \in V^{\mathcal{Y}}_n \Leftrightarrow \exists p(x \in V^{\mathcal{X}}_p \wedge A(p, n))$$
$$\varphi^{-1}(x) \in V^{\mathcal{X}}_n \Leftrightarrow \exists p(x \in V^{\mathcal{Y}}_p \wedge B(p, n))$$

We define the Σ^0_1 relations

$$S^{\mathcal{X}}(m, n) \Leftrightarrow \exists k, l \in \omega(A(m, k) \wedge S(k, l) \wedge B(l, n))$$
$$T^{\mathcal{X}}(m, n, i) \Leftrightarrow \exists k, l, j \in \omega(A(m, k) \wedge S(k, l) \wedge B(l, n) \wedge T(k, l, j) \wedge A(i, j))$$

We claim that $S^{\mathcal{X}}$ and $T^{\mathcal{X}}$ witness that $\mathcal{X}$ is recursively regular. Indeed:

1. We have

$$\begin{aligned}\exists m(x \in V^{\mathcal{X}}_m \wedge S^{\mathcal{X}}(m, n)) &\Leftrightarrow \exists m, k, l(x \in V^{\mathcal{X}}_m \wedge A(m, k) \wedge S(k, l) \wedge B(l, n)) \\ &\Leftrightarrow \exists k, l(\varphi(x) \in V^{\mathcal{Y}}_k \wedge S(k, l) \wedge B(l, n)) \\ (\mathcal{Y} \text{ recursively regular})- &\Leftrightarrow \exists l(\varphi(x) \in V^{\mathcal{Y}}_l \wedge B(l, n)) \\ &\Leftrightarrow x = \varphi^{-1}(\varphi(x)) \in V^{\mathcal{X}}_n\end{aligned}$$

2. We fix $m, n \in \omega$ such that $S^{\mathcal{X}}(m, n)$ holds and $k, l \in \omega$ such that $A(m, k) \wedge S(k, l) \wedge B(l, n)$ holds. We have:

$$\varphi[V^{\mathcal{X}}_m] \subseteq V^{\mathcal{Y}}_k \subseteq P_{k,l} \subseteq V^{\mathcal{Y}}_l \subseteq \varphi[V^{\mathcal{X}}_n]$$

where $P_{k,l} = \{y \in Y \mid \forall q(T(k, l, q) \Rightarrow y \notin V^{\mathcal{Y}}_q)\}$.
Now, let $P^{\mathcal{X}}_{m,n} = \{x \in X \mid \forall i(T^{\mathcal{X}}(m, n, i) \Rightarrow x \notin V^{\mathcal{X}}_i)\}$, observe that for any $j \in \omega$ the following holds

$$\exists i(x \in V^{\mathcal{X}}_i \wedge A(i, j)) \Leftrightarrow \varphi(x) \in V^{\mathcal{Y}}_j \Leftrightarrow x \in \varphi^{-1}[V^{\mathcal{Y}}_j]$$

and hence we get

$$\exists j(T(k,l,j) \wedge \exists i(x \in V_i^{\mathcal{X}} \wedge A(i,j))) \Leftrightarrow \varphi(x) \notin P_{k,l} \Leftrightarrow x \notin \varphi^{-1}[P_{k,l}]$$

the last equivalence is obvious, for the first one we explain both directions for clarity:

- $\Rightarrow$ Towards a contradiction assume that $\varphi(x) \in P_{k,l}$ and that the left-term holds. Then, by the first condition:

  $$(\varphi(x) \in P_{k,l} \wedge T(k,l,j)) \Rightarrow \varphi(x) \notin V_j^{\mathcal{Y}}$$

  but, for the second

  $$\exists i(x \in V_i^{\mathcal{X}} \wedge A(i,j)) \Rightarrow \varphi(x) \in V_j^{\mathcal{Y}}$$

  thus we reached a contradiction. ↯
- $\Leftarrow$ Given $\varphi(x) \notin P_{k,l}$, we have that $\exists j \in \omega$ such that $T(k,l,j)$ and $\varphi(x) \in V_j$ (because otherwise $\varphi(x)$ would be in $P_{k,l}$).

Therefore, we get

$$P_{m,n}^{\mathcal{X}} = \bigcap_{k,l \in \omega} \{\varphi^{-1}[P_{k,l}] \mid A(m,k) \wedge S(k,l) \wedge B(l,n)\}$$

indeed:

$$\begin{aligned} x \in P_{m,n}^{\mathcal{X}} &\Leftrightarrow \forall i(T^{\mathcal{X}}(m,n,i) \Rightarrow x \notin V_i^{\mathcal{X}}) \\ &\Leftrightarrow \forall i(\neg T^{\mathcal{X}}(m,n,i) \vee x \notin V_i^{\mathcal{X}}) \\ &\Leftrightarrow \forall i(\forall k,l,j(\neg A(m,k) \vee \neg S(k,l) \vee \neg B(l,n) \vee \neg T(k,l,j) \\ &\qquad \vee \neg A(i,j)) \vee x \notin V_i^{\mathcal{X}}) \\ &\Leftrightarrow \forall k,l((\neg A(m,k) \vee \neg S(k,l) \vee \neg B(l,n)) \\ &\qquad \vee \forall i,j(\neg T(k,l,j) \vee \neg A(i,j) \vee x \notin V_i^{\mathcal{X}})) \\ &\Leftrightarrow \forall k,l((A(m,k) \wedge S(k,l) \wedge B(l,n)) \Rightarrow x \in \varphi^{-1}[P_{k,l}]) \end{aligned}$$

We observe that:

- for any $k,l \in \omega$ such that $A(m,k) \wedge S(k,l) \wedge B(l,n)$ holds we get $V_m^{\mathcal{X}} \subseteq \varphi^{-1}[V_k^{\mathcal{X}}] \subseteq \varphi^{-1}[P_{k,l}]$, thus $V_m^{\mathcal{X}} \subseteq P_{m,n}^{\mathcal{X}}$.
- for any m,n such that $S^{\mathcal{X}}(m,n)$ holds, there are $k,l \in \omega$ satisfying $A(m,k) \wedge S(k,l) \wedge B(l,n)$ and such that $P_{m,n}^{\mathcal{X}} \subseteq \varphi^{-1}[P_{k,l}] \subseteq \varphi^{-1}[V_l^{\mathcal{Y}}] \subseteq V_n^{\mathcal{X}}$.

This concludes the proof because, given a recursive space $\mathcal{X}$ it is recursively isomorphic to a subspace of a recursively presented metric space (that is recursively regular by the first two steps). Hence it is recursively regular by the last step. □

Lemma 1.25 (Effective normality)**.** Given a recursively regular space $\mathcal{X}$, then:

1. If $P, Q \in \Pi^0_1(\mathcal{X})$ are disjoint, then there are two disjoint sets $P^*, Q^* \in \Sigma^0_1(\mathcal{X})$ such that $P \subseteq P^*$ and $Q \subseteq Q^*$.
2. The property 1. holds uniformly, that is there are computable functions $\varphi : \omega^2 \to \omega$ and $\psi : \omega^2 \to \omega$ such that, if

$$P = X \setminus \{x \in X \mid \exists p(x \in V^{\mathcal{X}}_p \wedge p \in W_n)\}$$
$$Q = X \setminus \{x \in X \mid \exists p(x \in V^{\mathcal{X}}_p \wedge p \in W_m)\}$$

are disjoint, then

$$P^* = \{x \in X \mid \exists p(x \in V^{\mathcal{X}}_p \wedge p \in W_{\varphi(n,m)})\}$$
$$Q^* = \{x \in X \mid \exists p(x \in V^{\mathcal{X}}_p \wedge p \in W_{\psi(n,m)})\}$$

are disjoint and, moreover, satisfy $P \subseteq P^*$ and $Q \subseteq Q^*$.

Proof. 1. Since $P, Q \in \Pi^0_1(\mathcal{X})$, then their complements are Σ^0_1 and hence there are $A, B \in \Sigma^0_1(\omega)$ such that

$$x \notin P \Leftrightarrow \exists p(x \in V^{\mathcal{X}}_p \wedge p \in A)$$
$$x \notin Q \Leftrightarrow \exists p(x \in V^{\mathcal{X}}_p \wedge p \in B)$$

we consider the relations S and T witnessing that $\mathcal{X}$ recursively regular, and define:

$$A'(q) \Leftrightarrow \exists p(p \in A \wedge S(q,p))$$
$$B'(q) \Leftrightarrow \exists p(p \in B \wedge S(q,p))$$

we notice that thanks to the properties of S

$$\begin{aligned} x \notin P &\Leftrightarrow \exists p(x \in V^{\mathcal{X}}_p \wedge p \in A) \\ &\Leftrightarrow \exists p \exists q(x \in V^{\mathcal{X}}_q \wedge p \in A \wedge S(q,p)) \\ &\Leftrightarrow \exists q(x \in V^{\mathcal{X}}_q \wedge q \in A') \end{aligned}$$

Moreover, since $A' \in \Sigma^0_1(\omega)$ there is a recursive $A^* \subseteq \omega^2$ such that $A'(q) \Leftrightarrow \exists m A^*(q,m)$. Similarly we get the same property for Q and define B^*. Now we define:

$$\begin{aligned} x \in P^* \Leftrightarrow \exists p \exists m \big(x \in V^{\mathcal{X}}_p \wedge B^*(p,m) \wedge \\ &\forall \langle q,n \rangle < \langle p,m \rangle (A^*(q,n) \Rightarrow \\ &\exists i \exists j (A(i) \wedge S(q,i) \wedge T(q,i,j) \wedge x \in V^{\mathcal{X}}_j))\big) \end{aligned}$$

and

$$x \in Q^* \Leftrightarrow \exists p \exists m \big(x \in V_p^{\mathcal{X}} \wedge A^*(p,m) \wedge \\ \forall \langle q,n \rangle \leq \langle p,m \rangle (B^*(q,n) \Rightarrow \\ \exists i \exists j (B(i) \wedge S(q,i) \wedge T(q,i,j) \wedge x \in V_j^{\mathcal{X}}))\big)$$

by their definitions $P^*, Q^* \in \Sigma_1^0(\mathcal{X})$. We claim that they verify the thesis. In particular, $P \subseteq P^*$ (and similarly $Q \subseteq Q^*$) because if $x \in P$, then $x \notin Q$, hence there is a $p \in \omega$ such that $x \in V_p^{\mathcal{X}}$ and $B'(p)$. Therefore, for some $q, m \in \omega$, $x \in V_q^{\mathcal{X}}$, $S(q,p)$ and $B(q,m)$.
We now pick any $\langle r,n \rangle < \langle q,m \rangle$ (actually this works for any $\langle r,n \rangle$), and suppose $A^*(r,n)$. Then for some $i \in \omega$, $A(i)$ and $S(r,i)$ hold, so $V_r^{\mathcal{X}} \subseteq V_i^{\mathcal{X}}$ and $V_i^{\mathcal{X}} \cap P = \varnothing$.
In particular, $x \notin V_i^{\mathcal{X}}$, so $x \notin P_{r,i}$ and hence there is some $j \in \omega$ such that $x \in V_j^{\mathcal{X}}$ and $T(r,i,j)$. This is exactly $x \in P^*$.
Finally, we need to prove that $P^* \cap Q^* = \varnothing$, towards a contradiction suppose that there is a $x \in P^* \cap Q^*$.
Then there are some $p, m, q, n \in \omega$ such that $B^*(p,m)$ and $A^*(q,n)$ hold. Suppose that $\langle q,n \rangle < \langle p,m \rangle$ (the case $\langle p,m \rangle \leq \langle q,n \rangle$ is similar). Then as before we get a pair $i, j \in \omega$ with $S(q,i)$, $T(q,i,j)$ and $x \in V_j^{\mathcal{X}}$. But as $S(q,i)$ and $T(q,i,j)$ imply $V_j^{\mathcal{X}} \cap V_q^{\mathcal{X}} = \varnothing$, then we have $x \notin V_q^{\mathcal{X}}$, a contradiction. ↯

2. The proof is basically the same of the previous point, the crucial point is that the definitions of P^* and Q^* are uniform. Said $W \in \omega^2$ the universal set for ω we define:

$$W'(n,p) \Leftrightarrow \exists q (W(n,q) \wedge S(p,q))$$

Therefore, if $A = W_n$ and $B = W_m$ (where A and B are as in the previous point) then $A' = W'_n$ and $B' = W'_m$. Moreover, if we consider a recursive $W^* \subseteq \omega^3$ such that $W'(n,p) \Leftrightarrow \exists l W^*(n,p,l)$ then $A^* = W^*_n$ and $B^* = W^*_m$. Now we define:

$$C^*(x,n,m) \Leftrightarrow \exists p \exists l \big(x \in V_p^{\mathcal{X}} \wedge W^*(m,p,l) \wedge \\ \forall \langle r,s \rangle < \langle p,l \rangle (W^*(n,r,s) \Rightarrow \\ \exists i \exists j (W(n,i) \wedge S(r,i) \wedge T(r,i,j) \wedge x \in V_j^{\mathcal{X}}))\big)$$

and

$$D^*(x,n,m) \Leftrightarrow \exists p \exists l \big(x \in V_p^{\mathcal{X}} \wedge W^*(n,p,l) \wedge \\ \forall \langle r,s \rangle \leq \langle p,l \rangle (W^*(m,r,s) \Rightarrow \\ \exists i \exists j (W(m,i) \wedge S(r,i) \wedge T(r,i,j) \wedge x \in V_j^{\mathcal{X}}))\big)$$

in this way $P^* = C^*_{n,m}$ and $Q^* = D^*_{n,m}$. Moreover, $C^*, D^* \in \Sigma^0_1(\mathcal{X} \times \omega^2)$ and hence there are $C, D \in \Sigma^0_1(\omega^3)$ such that

$$C^*(x,n,m) \Leftrightarrow \exists p(x \in V^{\mathcal{X}}_p \wedge C(n,m,p))$$
$$D^*(x,n,m) \Leftrightarrow \exists p(x \in V^{\mathcal{X}}_p \wedge D(n,m,p))$$

Now consider the computable function

$$h(n,m,p) = \begin{cases} 1 & \text{if } C(n,m,p) \\ \uparrow & \text{otherwise} \end{cases}$$

so $h = \varphi_e$ and hence by the S-m-n Theorem we consider $S^1_2 : \omega^3 \to \omega$ and hence:

$$\varphi_{S^1_2(e,n,m)}(p) = \varphi_e(n,m,p) = h(n,m,p)$$

In particular, $W_{S^1_2(e,n,m)} = C_{m,n}$ thus we define as $\varphi : \omega^2 \to \omega$ as $\varphi(m,n) = S^1_2(e,n,m)$. Similarly we can define $\psi : \omega^2 \to \omega$ for D and hence we get the thesis. □

We can iterate the process and, thanks to the uniformity of the construction, we get the following:

Theorem 1.26 (Effective Urysohn Theorem)**.** Given a recursively regular space $\mathcal{X}$, then:

1. If $P, Q \in \Pi^0_1(\mathcal{X})$ are disjoint, then there exists a Σ^0_1-recursive function $f : \mathcal{X} \to [0,1]$ such that $\forall x \in P(f(x) = 0)$ and $\forall x \in Q(f(x) = 1)$.
2. In fact, there is a computable function $\theta : \omega^2 \to \omega$ such that for all $n, m \in \omega$, if

$$P = X \setminus \{x \in X \mid \exists p(x \in V^{\mathcal{X}}_p \wedge p \in W_n)\}$$
$$Q = X \setminus \{x \in X \mid \exists p(x \in V^{\mathcal{X}}_p \wedge p \in W_m)\}$$

then $\theta(n,m)$ is the code of a Σ^0_1 subset of ω^2, and if we let

$$A_{n,m} = \left\{(x,p) \in X \times \omega \middle| \exists q \left(x \in V^{\mathcal{X}}_q \wedge (p,q) \in W^2_{\theta(n,m)}\right)\right\}$$

then there is a Σ^0_1-recursive function $f_{m,n} : X \to [0,1]$, such that $A_{n,m} = D_{f_{n,m}}$, $\forall x \in P(f_{n,m}(x) = 0)$ and $\forall x \in Q(f_{n,m}(x) = 1)$.

Proof. 1. Let $U_0 = \emptyset \subseteq P_0 = P \subseteq U_1 = X \setminus Q \subseteq P_1 = X$. We define inductively, on the dyadic numbers $r = k \cdot 2^{-n}$, with k odd less than 2^n and $n \geq 1$, a set $U_r \in \Sigma^0_1(\mathcal{X})$ and a set $P_r \in \Pi^0_1(\mathcal{X})$, using the computable functions $\varphi, \psi : \omega^2 \to \omega$ of the previous lemma, as follows: suppose that $P_{\frac{k}{2^n}} \subseteq U_{\frac{k+1}{2^n}}$ have been defined, define

$$U_{\frac{2k+1}{2^{n+1}}} := \left(P_{\frac{k}{2^n}}\right)^* \qquad P_{\frac{2k+1}{2^{n+1}}} := X \setminus \left(X \setminus U_{\frac{k+1}{2^n}}\right)^*$$

thanks to the functions φ and ψ such construction can be done uniformly and hence the relations (in $\mathcal{X} \times \omega^2$) $x \notin P_{\frac{k}{2^n}}$ and $x \in U_{\frac{k}{2^n}}$ are Σ^0_1. Moreover, for r, r' dyadic with $r < r'$, $U_r \subseteq P_r \subseteq U_{r'}$. Hence we define:

$$\begin{aligned} &f : \mathcal{X} \to [0,1] \\ &\quad x \mapsto \sup\{r \mid r = k \cdot 2^{-n} \wedge k \text{ odd } \wedge k < 2^n \wedge n \geq 1 \wedge x \notin U_r\} \end{aligned}$$

We observe that:

- $\forall x \in Q = X \setminus U_1 (f(x) = 1)$
- Since $P = P_0 \subseteq U_r$ for any such r, then $\forall x \in P (f(x) = 0)$
- Moreover

$$f(x) > q_l \Leftrightarrow \exists k, n \in \omega \left(k \text{ odd } \wedge \frac{k}{2^n} > q_l \wedge x \notin P_{\frac{k}{2^n}}\right)$$
$$f(x) < q_l \Leftrightarrow \exists k, n \in \omega \left(k \text{ odd } \wedge \frac{k}{2^n} < q_l \wedge x \in U_{\frac{k}{2^n}}\right)$$

thus $f : X \to [0,1]$ is recursive.

2. This point follows form the previous in a similar way as we did in Lemma 1.25, so we only sketch the proof: considered $P = W_n$ and $Q = W_m$, we can define the sequences (P_r, U_r) for all the dyadic numbers r as before, and define a set $A \in \Sigma^0_1(\omega^2 \times \mathcal{X} \times \omega)$ such that if P and Q are disjoint, $A(n, m, x, p)$ corresponds to the relation $q_{(p)_0} < f_{n,m}(x) < q_{(p)_1}$, where $f_{n,m}$ is the function defined in the point 1. Finally, applying the S-m-n Theorem in the same way as at the end of Lemma 1.25 there is a $\theta : \omega^2 \to \omega$ such that:

$$A(n, m, x, p) \Leftrightarrow \exists q (x \in V^{\mathcal{X}}_q \wedge W^2(\theta(n, m), p, q)) \qquad \square$$

Corollary 1.27. Every recursively regular space is recursively isomorphic to a subspace of the recursively presented Polish space $[0,1]^\omega$.

Proof. Let $h : \omega \to \omega$ be a computable function enumerating the set $\{\langle p, q\rangle \mid S(p, q)\}$, we define

$$A(n, i) \Leftrightarrow T((h(n))_0, (h(n))_1, i)$$
$$B(n, i) \Leftrightarrow i = (h(n))_1$$

in this way, for a fixed $n \in \omega$ with $h(n) = \langle p, q \rangle$, the Π^0_1 sets in $\mathcal{X}$ defined, respectively, by A_n and B_n are

$$X \setminus \{x \in X \mid \exists i(x \in V^{\mathcal{X}}_i \wedge A(n,i))\} = P_{p,q}$$
$$X \setminus \{x \in X \mid \exists i(x \in V^{\mathcal{X}}_i \wedge B(n,i))\} = X \setminus V^{\mathcal{X}}_q$$

Notice that these Π^0_1 sets are disjoint for any $n \in \omega$ (because $P_{p,q} \subseteq V^{\mathcal{X}}_q$). Moreover, since $A, B \in \Sigma^0_1(\omega^2)$, by the S-m-n Theorem there are two computable functions $s_A, s_B : \omega \to \omega$ such that

$$A(n,i) \Leftrightarrow W(s_A(n), i)$$
$$B(n,i) \Leftrightarrow W(s_B(n), i)$$

By the Effective Urysohn Theorem 1.26, for each $n \in \omega$ with $h(n) = \langle p, q \rangle$ the set $W^2_{\theta(s_A(n), s_B(n))}$ defines the diagram of a recursive function $f_n : \mathcal{X} \to [0,1]$ which is 0 on $P_{p,q}$ and 1 on $X \setminus V^{\mathcal{X}}_q$. Moreover, by the uniformity of the construction, since $W^2 \in \Sigma^0_1(\omega^3)$ and θ, s_A, s_B are computable, the function $f : \omega \times \mathcal{X} \to [0,1]$ defined by $f(n,x) = f_n(x)$ is Σ^0_1-recursive. We now define:

$$i : \mathcal{X} \to [0,1]^\omega$$
$$x \mapsto i(x) = (f_n(x))_{n \in \omega}$$

The function $i : \mathcal{X} \to [0,1]^\omega$ is Σ^0_1-recursive by Proposition 1.11, and, moreover, is injective[4]. Indeed, given $x_1, x_2 \in X$ with $x_1 \neq x_2$ there exists an $l \in \omega$ such that

$$(x_1 \in V^{\mathcal{X}}_l \wedge x_2 \notin V^{\mathcal{X}}_l) \dot{\vee} (x_1 \notin V^{\mathcal{X}}_l \wedge x_2 \in V^{\mathcal{X}}_l)$$

Suppose that the first case occurs (the other one is similar), then $x_1 \in V^{\mathcal{X}}_l$ implies that $\exists p(x \in V^{\mathcal{X}}_p \wedge S(p,l))$. Thus considered $m \in \omega$ such that $h(m) = \langle p, l \rangle$, and the corresponding function $f_m : \mathcal{X} \to [0,1]$ we have

- $\forall x \in P_{p,l}(f_m(x) = 0)$, and hence $f_m(x_1) = 0$
- $\forall x \in X \setminus V^{\mathcal{X}}_l(f_m(x) = 1)$, and hence $f_m(x_2) = 1$

This implies that $i(x_1) \neq i(x_2)$, and hence i is injective.
Finally, to conclude we have to prove that its inverse $j : i[X] \to \mathcal{X}$ is Σ^0_1-recursive. In particular, we claim that for any $\overline{y} = (y_n)_{n \in \omega} \in i[X]$

$$j(\overline{y}) \in V^{\mathcal{X}}_q \Leftrightarrow \exists p \exists n(S(p,q) \wedge h(n) = \langle p, q \rangle \wedge y_n < 1)$$

since the right term of the equivalence is Σ^0_1 in $[0,1]^\omega \times \omega$, this equivalence prove the recursivity of $j : i[X] \to \mathcal{X}$.

[4]In [Lou19, Section 2.4.9] the proof of the injectivity of $i : \mathcal{X} \to [0,1]^\omega$ is omitted, however this point is were we need the hypothesis of being T_0, indeed if the topology isn't able to distinguish two distinct points $x_1, x_2 \in X$, then $i(x_1) = i(x_2)$.

⇒ if $x = j(\overline{y}) \in V_q^{\mathcal{X}}$, then, since $\mathcal{X}$ recursively regular, there is a $p \in \omega$ with $S(p,q)$ and $x \in V_p^{\mathcal{X}}$. But by our construction, there is an $n \in \omega$ such that $h(n) = \langle p,q \rangle$ and the corresponding function $f_n : \mathcal{X} \to [0,1]$ is 0 on $P_{p,q}$, and hence on $V_p^{\mathcal{X}}$, so that $y_n = f_n(x) = f_n(j(\overline{y})) = 0$.

⇐ given $p,n \in \omega$ satisfying $h(n) = \langle p,q \rangle$ and $S(p,q)$, then $f_n : \mathcal{X} \to [0,1]$ is 1 on $X \setminus V_q^{\mathcal{X}}$, so as $y_n = f_n(j(\overline{y})) < 1$, one must have $j(\overline{y}) \in V_q^{\mathcal{X}}$. □

Proof of Theorem 1.23 - Left-to-right implication.
Thanks to the previous corollary, we have that any recursively regular space is recursively isomorphic to a subspace of $[0,1]^\omega$. Thus by definition, it is recursive. □

We will work from now on mainly with recursive spaces and recursively presented metric spaces, but thanks to Theorem 1.23, when useful we will use the corresponding semirecursive relations R, S and T that make them recursively regular when needed.

1.2 Topological and effective pointclasses

In this section, we define and prove the main properties of some collections of sets called pointclasses. But what is exactly a pointclass? To answer to this question we follow a similar approach to the one used in [CMM22, Section 3].

Definition 1.28. A **topological pointclass** (also called **boldface pointclass**) is a class-function $\mathbf{\Gamma}$ which satisfies the following requirements:

- The domain of $\mathbf{\Gamma}$ is the class of all topological spaces
- Given Z topological space, $\mathbf{\Gamma}(Z) \subseteq \mathcal{P}(Z)$.
- $\mathbf{\Gamma}$ is closed by continuous substitutions, that is: if $f : Z \to W$ is continuous and $A \in \mathbf{\Gamma}(W)$, then $f^{-1}[A] \in \mathbf{\Gamma}(Z)$.

Since we are interested in their effective counterpart, now that we have the class of Σ_1^0 sets and the Σ_1^0-recursivity, we define also:

Definition 1.29. An **effective pointclass** (also called **lightface pointclass**) is a class-function Γ which satisfies the following requirements:

- The domain of Γ is the class of all basic spaces
- Given $\mathcal{Z}$ basic space, $\Gamma(\mathcal{Z}) \subseteq \mathcal{P}(Z)$.

- Γ is closed by Σ^0_1-recursive substitutions, that is: if $f : \mathcal{Z} \to \mathcal{W}$ is Σ^0_1-recursive and $A \in \Gamma(\mathcal{W})$, then $f^{-1}[A] \in \Gamma(\mathcal{Z})$.

With a slight abuse of notation, given a topological pointclass Γ and a set A we write $A \in \Gamma$ if $A \in \Gamma(X)$ for some topological space X (and similarly for the effective pointclasses). Moreover, to each pointclass we associate:

- the **dual pointclass** $\neg\Gamma = \{\neg A \mid A \in \Gamma\}$
- the **ambiguous pointclass** $\Delta(\Gamma) = \Gamma \cap \neg\Gamma$
- $\exists^0\Gamma = \{\exists^0 A \mid A \in \Gamma\}$ where, for $A \subseteq \mathcal{X} \times \omega$:

  $$x \in \exists^0 A \Leftrightarrow \exists m \in \omega(A(x,m))$$

  and $\forall^0\Gamma$ is defined similarly
- $\exists^1\Gamma = \{\exists^1 A \mid A \in \Gamma\}$ where, for $A \subseteq \mathcal{X} \times \omega^\omega$:

  $$x \in \exists^1 A \Leftrightarrow \exists \alpha \in \omega^\omega(A(x,\alpha))$$

  and $\forall^1\Gamma$ is defined similarly.

An example of topological pointclass is the class-function **Bor** that associate to each topological space (X, τ_X) the collection of all Borel subsets of X, that is the smallest collection of subsets of X containing τ_X and closed under complements and countable unions.
From now on, for a generic boldface pointclass we use the boldsymbol $\boldsymbol{\Gamma}$, while for a generic lightface pointclass or for a property that is defined for both we use the symbol Γ.

1.2.1 The Borel and Projective hierarchies

Definition 1.30 (Borel Hierarchy)**.** Let (X, τ_X) be a topological space. The **Borel hierarchy** on X is defined by (simultaneous) induction on the positive ordinals $\alpha \leq \omega_1$ as follows:

$$\boldsymbol{\Sigma}^0_1(X) = \tau_X = \{A \mid A \text{ open}\} \qquad \boldsymbol{\Pi}^0_1(X) = \{A \mid A \text{ closed}\}$$

$$\boldsymbol{\Sigma}^0_\alpha(X) = \left\{ \bigcup_{n\in\omega} A_n \,\middle|\, A_n \in \bigcup_{1\leq\beta<\alpha} \boldsymbol{\Pi}^0_\beta \right\} \quad \boldsymbol{\Pi}^0_\alpha(X) = \left\{ \bigcap_{n\in\omega} A_n \,\middle|\, A_n \in \bigcup_{1\leq\beta<\alpha} \boldsymbol{\Sigma}^0_\beta \right\}$$

$$\boldsymbol{\Delta}^0_\alpha(X) = \Delta(\boldsymbol{\Sigma}^0_\alpha(X))$$

Definition 1.31 (Projective Hierarchy)**.** Let (X, τ_X) be a separable metrizable space. The **Projective hierarchy** on X is defined by (simultaneous) induction on the natural numbers:

$$\boldsymbol{\Sigma}^1_1(X) = \exists^1 \boldsymbol{\Pi}^0_1(X \times \omega^\omega) \qquad \boldsymbol{\Pi}^1_1(X) = \neg\boldsymbol{\Sigma}^1_1(X)$$

$$\boldsymbol{\Sigma}^1_{n+1}(X) = \exists^1 \boldsymbol{\Pi}^1_n(X \times \omega^\omega) \quad \boldsymbol{\Pi}^1_{n+1}(X) = \neg\boldsymbol{\Sigma}^1_{n+1}(X) \quad \boldsymbol{\Delta}^1_n(X) = \Delta(\boldsymbol{\Sigma}^1_n(X))$$

We recall some classical properties of these pointclasses.

Proposition 1.32 ([Kec95, Propositions 22.1 and 37.1])**.** Given X metrizable space and a positive ordinal $\alpha \leq \omega_1$.

1. $\boldsymbol{\Pi}^0_\alpha(X) = \neg\, \boldsymbol{\Sigma}^0_\alpha(X)$
2. $\mathbf{Bor}(X) = \bigcup_{1\leq\alpha<\omega_1} \boldsymbol{\Sigma}^0_\alpha(X) = \bigcup_{1\leq\alpha<\omega_1} \boldsymbol{\Pi}^0_\alpha(X) = \bigcup_{1\leq\alpha<\omega_1} \boldsymbol{\Delta}^0_\alpha(X)$
3. Any Borel or projective pointclass is closed under continuous substitution (and hence is a topological pointclass), finite conjunction, finite disjunction and bounded quantifications on ω.
4. $\boldsymbol{\Sigma}^0_\alpha$ is closed under countable union, $\boldsymbol{\Pi}^0_\alpha$ is closed under countable intersections, and $\boldsymbol{\Delta}^0_\alpha$ is closed under complements.
5. Any projective pointclass is closed countable union and countable intersection.
6. $\boldsymbol{\Sigma}^1_n$ is closed under continuous images, and $\boldsymbol{\Delta}^1_n$ is closed under complements.

Theorem 1.33 (Inclusion diagram [Kec95, Section 11.B])**.** Given X Polish space we have:

$$\begin{array}{ccccccccccccc}
 & \boldsymbol{\Sigma}^0_1(X) & & \boldsymbol{\Sigma}^0_2(X) & & & \cdots & & \boldsymbol{\Sigma}^1_1(X) & & & \cdots \\
 \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} \ \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \\
\boldsymbol{\Delta}^0_1(X) & & \boldsymbol{\Delta}^0_2(X) & & \boldsymbol{\Delta}^0_3(X) & & \cdots & \boldsymbol{\Delta}^1_1(X) & & \boldsymbol{\Delta}^1_2(X) & & \cdots \\
 \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} \ \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \\
 & \boldsymbol{\Pi}^0_1(X) & & \boldsymbol{\Pi}^0_2(X) & & & \cdots & & \boldsymbol{\Pi}^1_1(X) & & & \cdots
\end{array}$$

The part of the diagram regarding the Borel pointclasses holds also for any metrizable space. Moreover, such inclusions are strict for uncountable spaces (see [Kec95, Theorem 22.4]).

1.2.2 The Kleene's hierarchy

We inductively define the *Kleene's pointclasses*.

Definition 1.34 (Arithmetical Hierarchy)**.** The pointclasses in the Arithmetical Hierarchy are defined by (simultaneous) induction on the natural numbers:

$$\Sigma^0_1 = \text{effectively open sets} \quad \Pi^0_1 = \neg\Sigma^0_1$$
$$\Sigma^0_{n+1} = \exists^0\Pi^0_n \quad \Pi^0_{n+1} = \neg\Sigma^0_{n+1} \quad \Delta^0_n = \Delta(\Sigma^0_n)$$

Definition 1.35 (Analytical Hierarchy)**.** The pointclasses in the Analytical Hierarchy are defined by (simultaneous) induction on the natural numbers:

$$\Sigma^1_1 = \exists^1 \Pi^0_1 \quad \Pi^1_1 = \neg\Sigma^1_1$$
$$\Sigma^1_{n+1} = \exists^1 \Pi^1_n \quad \Pi^1_{n+1} = \neg\Sigma^1_{n+1} \quad \Delta^1_n = \Delta(\Sigma^1_n)$$

We refer to all pointclasses in the Arithmetical Hierarchy or in the Analytical Hierarchy as the Kleene's pointclasses.
Given a function $\alpha : \omega \to \omega$ we can define the relativized-to-α lightface hierarchies considering α-semirecursive sets instead of semirecursive sets on ω in the definition of the Σ^0_1, and then define the whole hierarchy considering the pointclass $\Sigma^{0,\alpha}_1$ as starting point. Given a lightface pointclass Γ we denote with Γ^α the relativized-to-α pointclass.

Proposition 1.36 ([Lou19, Propositions 3.2.2. and 3.2.4.])**.**

1. Any Kleene's pointclass:
   (a) is closed under finite conjuctions, finite disjunctions and bounded quantifications on ω
   (b) is closed under recursive substitutions (and hence is an effective pointclass), fixing of integers arguments and recursive arguments.
2. Σ^0_n is closed under existential quantification on ω $\exists^0$, Π^0_n is closed under universal quantification on ω $\forall^0$, and Δ^0_n is closed under negation $\neg$.
3. Σ^1_n is closed under existential quantification on ω^ω $\exists^1$, Π^1_n is closed under universal quantification on ω^ω $\forall^1$, and Δ^1_n under negation $\neg$. Moreover, all the analytical pointclasses are closed under existential quantification on ω $\exists^0$ and universal quantification on ω $\forall^0$.

Theorem 1.37 (Inclusion diagram [Lou19, Theorem 3.2.3.])**.** Given $\mathcal{X}$ recursive space we have:

$$\begin{array}{ccccccccccc}
 & \Sigma^0_1(\mathcal{X}) & & \Sigma^0_2(\mathcal{X}) & & \cdots & & \Sigma^1_1(\mathcal{X}) & & \cdots \\
\Delta^0_1(\mathcal{X}) & & \Delta^0_2(\mathcal{X}) & & \Delta^0_3(\mathcal{X}) & \cdots & \Delta^1_1(\mathcal{X}) & & \Delta^1_2(\mathcal{X}) & \cdots \\
 & \Pi^0_1(\mathcal{X}) & & \Pi^0_2(\mathcal{X}) & & \cdots & & \Pi^1_1(\mathcal{X}) & & \cdots
\end{array}$$

Proof. We only prove that $\Sigma^0_1(\mathcal{X}) \subseteq \Sigma^0_2(\mathcal{X})$, the other inclusions follow by induction. We consider the semirecursive predicate S and T witnessing that

$\mathcal{X}$ is a recursively regular space, and $\forall i,j \in \omega$ such that $S(i,j)$ we consider $P_{i,j} \in \Pi^0_1$ such that $V_i^{\mathcal{X}} \subseteq P_{i,j} \subseteq V_j^{\mathcal{X}}$. Therefore, given $A \in \Sigma^0_1(\mathcal{X})$ with the associated $A^* \in \Sigma^0_1(\omega)$:

$$\begin{aligned} x \in A &\Leftrightarrow \exists p \in \omega(x \in V_p^{\mathcal{X}} \wedge A^*(p)) \\ &\Leftrightarrow \exists p,q \in \omega(x \in P_{q,p} \wedge \underbrace{S(q,p) \wedge A^*(p))}_{\Sigma^0_1(\omega^2)} \end{aligned}$$

(the last equivalence holds by the properties of S). Observe that $P_{p,q} \in \Pi^0_1(\mathcal{X})$ and that the other conditions are semirecursive, hence $A \in \Sigma^0_2(\mathcal{X})$ thanks to the closure properties under recursive substitutions and $\exists^0$. □

1.2.3 Relativization and universal sets

From the definitions it is clear that:

Proposition 1.38. Given $\mathcal{X}$ recursive space and a Kleene's pointclass Γ between Σ^0_n, Π^0_n, Σ^1_1 and Π^1_1, then said $\mathbf{\Gamma}$ the corresponding boldface pointclass we have that $\Gamma(\mathcal{X}) \subseteq \mathbf{\Gamma}(X)$ and for any $\alpha \in \omega^\omega$ $\Gamma^\alpha(\mathcal{X}) \subseteq \mathbf{\Gamma}(X)$.

The previous result shows that for any Kleene's pointclass $\bigcup_{\alpha\in\omega^\omega} \Gamma^\alpha(\mathcal{X}) \subseteq \mathbf{\Gamma}(X)$. Actually, we will see that the previous inclusion is an equality, that is $\bigcup_{\alpha\in\omega^\omega} \Gamma^\alpha(\mathcal{X}) = \mathbf{\Gamma}(X)$.

Definition 1.39. Given X and Y topological spaces and $\mathbf{\Gamma}$ boldface pointclass, $G \subseteq X \times Y$ **parametrizes** $\mathbf{\Gamma}(Y)$ if

$$\forall P \subseteq Y(P \in \mathbf{\Gamma} \Leftrightarrow \exists x \in X(P = G_x))$$

where $G_x = \{y \in Y \mid G(x,y)\}$ is called x-section of G. G is said **universal** for $\mathbf{\Gamma}(Y)$ if it parametrizes $\mathbf{\Gamma}(Y)$ and, in addition, it is in $\mathbf{\Gamma}$.

We can give the same definition for lightface pointclasses and recursive spaces.

Theorem 1.40 (Universal set for Kleene's pointclasses [Lou19, Theorem 3.3.5.])**.** Given $\mathcal{X}$ recursive space and a Kleene's pointclass Γ between Σ^0_n, Π^0_n, Σ^1_1 and Π^1_1, then:

1. There is a set $G \in \Gamma(\omega \times \mathcal{X})$ which is universal for $\Gamma(\mathcal{X})$.
2. The relativized result also holds uniformly, that is: there is a set $G \in \Gamma(\omega^\omega \times \omega \times \mathcal{X})$ such that for every $\alpha \in \omega^\omega$ its section $G_\alpha \in \Gamma^\alpha(\omega \times \mathcal{X})$ is universal for $\Gamma^\alpha(\mathcal{X})$.

Proof. 1. We have already proven the result for Σ^0_1 sets (Theorem 1.8). To prove it for all selected lightface classes, it is enough to observe that:

(a) If $H \subseteq \omega \times \mathcal{X}$ is universal for $\Gamma(\mathcal{X})$, then its complement $G = \neg H$ is universal for the dual class $\neg\Gamma$.

(b) If $H \subseteq \omega \times (\omega \times X)$ is universal for $\Gamma(\omega \times \mathcal{X})$, then G defined by:

$$(n, x) \in G \Leftrightarrow \exists m \in \omega(n, (m, x)) \in H$$

is universal for $\exists^0\Gamma(\mathcal{X})$

(c) Similarly, if $H \subseteq \omega \times (\omega^\omega \times X)$ is universal for $\Gamma(\omega^\omega \times \mathcal{X})$, then G defined by:

$$(n, x) \in G \Leftrightarrow \exists \alpha \in \omega^\omega(n, (\alpha, x)) \in H$$

is universal for $\exists^1\Gamma(\mathcal{X})$

2. The proof is identical considering the relativized halting set for the case for $\Sigma^0_1(\alpha)$ sets (the uniformity follows from the fact that the relativized Kleene's Enumeration Theorem can be proven uniformly in the oracles). □

Using this result, we can consider a $\mathbf{\Gamma}$-universal set $\tilde{G} \in \Gamma(\omega^\omega \times X)$. In particular, defining:

$$\tilde{G}(e^\frown\alpha, x) \Leftrightarrow G(\alpha, e, x)$$

where G is the universal set given by the second point of the previous theorem, we obtain such universal. Thanks to this we get: $\bigcup_{\alpha\in\omega^\omega} \Gamma^\alpha(\mathcal{X}) = \mathbf{\Gamma}(X)$.

Theorem 1.41 (Existence of good universal sets [Lou19, Theorem 3.3.8.])**.** Given $\mathcal{X}$ recursive space, and Γ Kleene's pointclass between Σ^0_n, Π^0_n, Σ^1_1 and Π^1_1, then:

1. there exists $W^\Gamma \in \Gamma(\omega\times\mathcal{X})$, such that for any $A \in \Gamma(\omega\times\mathcal{X})$ there exists an injective total recursive function $f_A : \omega \to \omega$:

$$\forall n \in \omega \forall x \in X(A(n, x) \Leftrightarrow W^\Gamma(f_A(n), x))$$

2. there exists $\boldsymbol{W}^\Gamma \in \Gamma(\omega^\omega \times \mathcal{X})$, such that for any $A \in \mathbf{\Gamma}(\omega^\omega \times X)$ there exists an injective total continuous function $f_A : \omega^\omega \to \omega^\omega$:

$$\forall \alpha \in \omega^\omega \forall x \in X(A(\alpha, x) \Leftrightarrow \boldsymbol{W}^\Gamma(f_A(\alpha), x))$$

Proof. 1. Let $G \in \Gamma(\omega \times (\omega \times \mathcal{X}))$ be the universal set for $\Gamma(\omega \times \mathcal{X})$ of Theorem 1.40, we define:

$$\forall n \in \omega \forall x \in X (W^\Gamma(n,x) \Leftrightarrow G((n)_0, ((n)_1, x)))$$

Given $A \in \Gamma(\omega \times \mathcal{X})$, then $A = G_a$ for some $a \in \omega$, then:

$$A(n,x) \Leftrightarrow G(a,(n,x)) \Leftrightarrow W^\Gamma(\langle a,n\rangle, x)$$

hence the desired function is $f_A(n) = \langle a,n\rangle$.
We finally observe that the set W^Γ satisfying the thesis is actually universal for $\Gamma(\omega^\omega \times \mathcal{X})$, indeed given $B \in \Gamma(\mathcal{X})$ we consider $A \in \Gamma(\omega \times \mathcal{X})$ defined by $A = \{(n,x) \mid B(x)\}$, and apply the result to A.

2. Let $\tilde{G} \in \Gamma(\omega^\omega \times (\omega^\omega \times \mathcal{X}))$ be the universal set for $\mathbf{\Gamma}(\omega^\omega \times X)$ obtained from the previous remark, we define:

$$\forall n \in \omega \forall x \in X (\boldsymbol{W}^\Gamma(\alpha,x) \Leftrightarrow \tilde{G}((\alpha)_0, ((\alpha)_1, x)))$$

where $()_0, ()_1 : \omega^\omega \to \omega^\omega$ are the continuous functions defined by $\alpha = (\alpha)_0 \oplus (\alpha)_1$. Now, given $A \in \mathbf{\Gamma}(\omega^\omega \times X)$, then $A = \tilde{G}_\alpha$ for some $\alpha \in \omega^\omega$, then:

$$A(\beta,x) \Leftrightarrow \tilde{G}(\alpha,(\beta,x)) \Leftrightarrow W^\Gamma(\alpha \oplus \beta, x)$$

hence the desired function is $f_A(\beta) = \alpha \oplus \beta$. Again the universality follows trivially. $\square$

The sets W^Γ and $\boldsymbol{W}^\Gamma$ are called *good universal sets* for Γ and $\mathbf{\Gamma}$ (respectively).

1.2.4 Parametrization systems

Definition 1.42. A **parametrization system** for a boldface pointclass $\mathbf{\Gamma}$ is a class-function $Y \to G^Y \subseteq \omega^\omega \times Y$ such that G^Y parametrizes $\mathbf{\Gamma}(Y)$ for every separable metrizable space Y. Similarly, an **universal system** is defined by requiring also that $G^Y \in \mathbf{\Gamma}(\omega^\omega \times Y)$.

Since Effective Descriptive Set Theory is usually developed for recursively presented Polish spaces, the definition of *parametrization system* is usually given for Polish spaces (see for example [GKN21, page 5]). However, since we extended the framework to an effective equivalent of second countable metrizable space we adapt it.

Definition 1.43. A parametrization system $(G^{\mathcal{Y}})_Y$ is

- **effectively good** if for every $\mathcal{X}$ of type 0, and every separable metrizable space Y exists a recursive function $S : \omega \times X \to \omega$ such that:

$$\forall e \in \omega \forall x \in X \forall y \in Y (G^{X\times Y}(e,x,y) \Leftrightarrow G^Y(S(e,x),y))$$

- **good** if for every $\mathcal{X}$ of type 1, and every separable metrizable space Y exists a continuous function $S : \omega^\omega \times X \to \omega^\omega$ such that:

$$\forall \varepsilon \in \omega^\omega \forall x \in X \forall y \in Y (G^{X\times Y}(\varepsilon,x,y) \Leftrightarrow G^Y(S(\varepsilon,x),y))$$

Again, one can give a similar definitions (as parametrization and universal system) considering lightface pointclasses and recursive spaces.
The good universal sets W^Γ and $\boldsymbol{W}^\Gamma$ introduced in Theorem 1.41 correspond to a good universal system for recursive spaces (we will see at the end of the chapter how to extend the same result to all separable metrizable spaces).

Proposition 1.44. Given a recursive space $\mathcal{Y}$, and Γ Kleene's pointclass between Σ^0_n, Π^0_n, Σ^1_1 and Π^1_1, then:

1. for $\mathcal{X}$ of type 0, there exists an injective recursive function $S : \omega\times\mathcal{X} \to \omega$ such that:

$$\forall e \in \omega \forall x \in X \forall y \in Y (W^{\Gamma,\mathcal{X}\times\mathcal{Y}}(e,x,y) \Leftrightarrow W^{\Gamma,\mathcal{Y}}(S(e,x),y))$$

2. for $\mathcal{X}$ of type 1, there exists an injective continuous function $S : \omega^\omega \times \mathcal{X} \to \omega^\omega$ such that:

$$\forall \varepsilon \in \omega^\omega \forall x \in X \forall y \in Y (\boldsymbol{W}^{\Gamma,\mathcal{X}\times\mathcal{Y}}(\varepsilon,x,y) \Leftrightarrow \boldsymbol{W}^{\Gamma,\mathcal{Y}}(S(\varepsilon,x),y))$$

Proof. 1. Suppose that $\mathcal{X} = \omega^k$, we define $A \in \Gamma(\omega \times \mathcal{X})$ as:

$$\forall m \in \omega \forall x \in X ((m,x) \in A \Leftrightarrow ((m)_0,(m)_1,\dots,(m)_k,x) \in W^{\Gamma,\mathcal{X}\times\mathcal{Y}})$$

hence we consider the injective function $f_A : \omega \to \omega$ corresponding to A given by Theorem 1.41. Hence:

$$\begin{aligned}(m_0,\dots,m_k,x) \in W^{\Gamma,\mathcal{X}\times\mathcal{Y}} &\Leftrightarrow (\langle m_0,\dots,m_k\rangle,x) \in A \\ &\Leftrightarrow (f_A(\langle m_0,\dots,m_k\rangle),x) \in W^{\Gamma,\mathcal{Y}}\end{aligned}$$

that gives our function S.

2. Suppose that $\mathcal{X} = \omega^k \times (\omega^\omega)^l$, we define $A \in \Gamma(\omega^\omega \times \mathcal{X})$ as:

$$\forall m_1^\frown \dots ^\frown m_k^\frown \alpha \in \omega^\omega \forall x \in X$$
$$((m_1^\frown \dots ^\frown m_k^\frown \alpha, x) \in A \Leftrightarrow ((\alpha)_0, m_1, \dots, m_k, (\alpha)_1, \dots, (\alpha)_l, x) \in \boldsymbol{W}^{\Gamma, \mathcal{X} \times \mathcal{Y}})$$

as in the previous case, we consider the injective function $f_A : \omega^\omega \to \omega^\omega$ given by Theorem 1.41 and hence:

$$\begin{aligned}(\beta_0, m_1, \dots, m_k, \beta_1, \dots, \beta_l, x) \in \boldsymbol{W}^{\Gamma, \mathcal{X} \times \mathcal{Y}} \\ \Leftrightarrow (m_1^\frown \dots m_k^\frown \beta_0 \oplus \dots \oplus \beta_l, x) \in A \\ \Leftrightarrow (f_A(m_1^\frown \dots m_k^\frown \beta_0 \oplus \dots \oplus \beta_l), x) \in \boldsymbol{W}^{\Gamma, \mathcal{Y}} \square\end{aligned}$$

Although we have built good universal sets, the property of being good or effectively good for a parametrization system is more general. We will see in the last section that we can create a good universal system for second countable metrizable spaces that is not made of good universal sets.

1.2.5 $\boldsymbol{\Gamma}$-measurable and Γ-recursive functions

Of particular interest in Classical Descriptive Set Theory (and for our work) are the Borel measurable functions. However, one can give a more general definition to extend the same concept to boldface pointclasses.

Definition 1.45. Given a bolface pointclass $\boldsymbol{\Gamma}$, and X, Y topological spaces a function $f : X \to Y$ is **$\boldsymbol{\Gamma}$-measurable** if $f^{-1}[U] \in \boldsymbol{\Gamma}(X)$ for any open set U of Y.

Similarly, we extend the concept of Σ^0_1-recursive function between recursive spaces to other lightface pointclasses. Indeed, as the former was an effective refinement of the continuity the latter is an effective version of the $\boldsymbol{\Gamma}$-measurability defined above.

Definition 1.46. Given $\mathcal{X}$ and $\mathcal{Y}$ recursive spaces, and Γ pointclass, a function $f : \mathcal{X} \to \mathcal{Y}$ is **Γ-recursive** if its diagram is in Γ, that is:

$$D_f = \{(x, n) \mid f(x) \in V_n^{\mathcal{Y}}\} \in \Gamma(\mathcal{X} \times \omega)$$

Notice that, for a lightface pointclass Γ, the definition of Γ-recursivity implies that for each $n \in \omega$ $f^{-1}[V_n^{\mathcal{Y}}] \in \Gamma(\mathcal{X})$ and hence, if the corresponding boldface pointclass $\boldsymbol{\Gamma}$ is closed under countable unions, a Γ-recursive function is also $\boldsymbol{\Gamma}$-measurable. Moreover, in general there is no requirement on the fact that Γ is lightface.

Theorem 1.47 (Dellacherie [Mos09, Lemma 3D.1]). Let Γ be a pointclass that contains Σ^0_1, closed under recursive substitutions (or continuous substitutions if it is boldface), finite conjunctions, finite disjunctions, bounded quantifications and $\exists^0$, then a function between recursive spaces $f : \mathcal{X} \to \mathcal{Y}$ is Γ-recursive if and only if for every $P \in \Sigma^0_1(\omega \times \mathcal{Y})$, then $P_f \coloneqq \{(n,x) \mid (n, f(x)) \in P\} \in \Gamma(\omega \times \mathcal{X})$.

Proof. $\Rightarrow$ Having $P \in \Sigma^0_1(\omega \times \mathcal{Y})$ means that there exists a $P^* \in \Sigma^0_1(\omega^2)$ such that:

$$(m,y) \in P \Leftrightarrow \exists n \in \omega(y \in V^{\mathcal{Y}}_n \wedge (m,n) \in P^*)$$

therefore:

$$P_f(m,x) \Leftrightarrow \exists n \in \omega(\underbrace{f(x) \in V^{\mathcal{Y}}_n}_{\in \Gamma} \wedge (m,n) \in P^*)$$

and hence $P_f \in \Gamma(\omega \times \mathcal{X})$ by the closure properties required.

$\Leftarrow$ It is immediate considering $P = \{(n,y) \mid y \in V^{\mathcal{Y}}_n\}$. □

The requirements for the pointclasses in the previous theorem are satisfied by the pointclasses Σ^0_n, Σ^1_n, Δ^1_n, Π^1_n and their corresponding boldface versions.

Proposition 1.48 ([Mos09, Exercise 3D.22]). Given a topological pointclass $\mathbf{\Gamma}$ as in the previous theorem but also closed under countable disjunctions and a function between recursive spaces $f : \mathcal{X} \to \mathcal{Y}$, then

$$f : \mathcal{X} \to \mathcal{Y}\ \mathbf{\Gamma}\text{-recursive} \Leftrightarrow f : X \to Y\ \mathbf{\Gamma}\text{-measurable}$$

Proof. $\Rightarrow$ Given U open in Y we have that $U = \bigcup_{i\in\omega} V^{\mathcal{Y}}_{u(i)}$ for some function $u : \omega^\omega \to \omega^\omega$. For every $n \in \omega$

$$f^{-1}[V^{\mathcal{Y}}_n] = \{x \in X \mid (x,n) \in D_f\} \in \mathbf{\Gamma}(X)$$

By the closure w.r.t. countable disjunction we have $f^{-1}[U] \in \mathbf{\Gamma}(X)$.

$\Leftarrow$ Given any $P \in \Sigma^0_1(\omega \times \mathcal{Y})$, said P_n the sections of P on the first coordinate we have:

$$P_f = \{(n,x) \mid (n, f(x)) \in P\} = \bigcup_{n\in\omega}(\{n\} \times f^{-1}[P_n])$$

that is in $\mathbf{\Gamma}$ by the required closure properties, and hence we conclude applying the previous theorem. □

Therefore, for the pointclasses $\mathbf{\Sigma}^0_n$, $\mathbf{\Sigma}^1_n$, $\mathbf{\Delta}^1_n$, and $\mathbf{\Pi}^1_n$, the concepts of $\mathbf{\Gamma}$-recursivity and $\mathbf{\Gamma}$-measurability coincide.

Lemma 1.49 ([Mos09, Exercise 3D.21])**.** Given $\mathcal{X}$ and $\mathcal{Y}$ recursive spaces and a function $f : \mathcal{X} \to \mathcal{Y}$, then:

$$\exists \alpha \in \omega^\omega (f : \mathcal{X} \to \mathcal{Y}\ \Sigma_1^{0,\alpha}\text{-recursive}) \Leftrightarrow f : X \to Y \text{ continuous}$$

Proof. $\Rightarrow$ Having f $\Sigma_1^{0,\alpha}$-recursive means that its diagram is semirecursive:

$$D_f = \{(x,n) \mid f(x) \in V_n^{\mathcal{Y}}\} \in \Sigma_1^{0,\alpha}(\mathcal{X} \times \omega)$$

In particular this implies that:

$$f^{-1}[V_n^{\mathcal{Y}}] = \{x \in X \mid (x,n) \in D_f\} \in \Sigma_1^{0,\alpha}(\mathcal{X})$$

therefore for any open $U = \bigcup_{i\in\omega} V_{u(i)}^{\mathcal{Y}}$ (where $u : \omega \to \omega$ enumerates the basic open forming U) we have:

$$f^{-1}[U] = f^{-1}\left[\bigcup_{i\in\omega} V_{u(i)}^{\mathcal{Y}}\right] = \bigcup_{i\in\omega} f^{-1}\left[V_{u(i)}^{\mathcal{Y}}\right] \in \mathbf{\Sigma}_1^0$$

$\Leftarrow$ The diagram of a continuous function is open, indeed:

$$D_f = \{(x,n) \mid f(x) \in V_n^{\mathcal{Y}}\} = \bigcup_{n\in\omega} f^{-1}\left[V_n^{\mathcal{Y}}\right] \times \{n\} \in \mathbf{\Sigma}_1^0(\mathcal{X} \times \omega)$$

As any open set is $\Sigma_1^{0,\alpha}$ with respect to some $\alpha \in \omega^\omega$ we get the thesis. □

1.3 Effectivity on arbitrary second countable metrizable spaces

1.3.1 Relativized recursively regular spaces and recursive spaces

We now introduce the relativized version of the Louveau's framework. We don't know if there are any other approaches similar to this, but, as shown in the remaining of the section, this seems to be quite natural. Moreover, it is coherent to the one used in [Mos09].

We recall from [Mos09, Section 3I] the definition of ε-recursively presented metric space:

Definition 1.50. Let (X, d) be a separable metric space and $\boldsymbol{r} = (r_i)_{i\in\omega}$ an enumeration (possibly with repetitions) of a dense subset of X. For $\varepsilon \in \omega^\omega$, we say that $\boldsymbol{r}$ is a ε**-recursive presentation** of X if the relations on ω^3

$$P(i,j,k) \Leftrightarrow \mathrm{d}(r_i, r_j) \le q_k$$
$$Q(i,j,k) \Leftrightarrow \mathrm{d}(r_i, r_j) < q_k$$

are ε-recursive. The structure $(X, \mathrm{d}, \boldsymbol{r})$ is called **ε-recursively presented metric space**. If moreover (X, d) is complete, then $(X, \mathrm{d}, \boldsymbol{r})$ is called **ε-recursively presented Polish space**.

Remark 1.51. It is clear that every separable metric space has an ε-recursive presentation, taking $\varepsilon \in \omega$ a sufficiently strong oracle to code the relations P and Q relative to a chosen dense set $\boldsymbol{r}$.

Following the proof in the first step of the right-to-left implication of Theorem 1.23, one can check that given $\mathcal{X}$ ε-recursively presented metric space the predicate R witnessing that $\mathcal{X}$ is a basic space, and the predicates S and T witnessing that $\mathcal{X}$ is recursively regular are in $\Sigma_1^{0,\varepsilon}$.
Moreover, as in the third step of the same proof, the same remark holds for spaces that are $\Sigma_1^{0,\varepsilon}$-recursively isomorphic to a subspace of an ε-recursively presented metric space.
This remark leads us to introduce the definitions of ε-basic space, ε-recursively regular space and ε-recursive space. The first two definitions are identical to the unrelativized version but require that the relations R, S, and T are in $\Sigma_1^{0,\varepsilon}$. However the latter is:

Definition 1.52. For $\varepsilon \in \omega^\omega$, an ε-basic space $\mathcal{X}$ is said:

- **ε-recursive** if it is $\Sigma_1^{0,\varepsilon}$-recursively isomorphic to a subspace of an ε-recursively presented metric space.
- **Polish ε-recursive** if it is $\Sigma_1^{0,\varepsilon}$-recursively isomorphic to an ε-recursively presented Polish space.

Notice that requiring only that the space $\mathcal{X}$ is recursively isomorphic to an ε-recursively presented metric space, is not a good way to extend the definition if we want to maintain the effective equivalent of Theorem 1.23. Indeed, considering the proof from left-to-right of this theorem, any ε-recursively regular space is $\Sigma_1^{0,\varepsilon}$-recursively isomorphic to a subspace of the Hilbert cube $[0,1]^\omega$.
We observe that all the results stated so far can be easily relativized and extended to this framework. Moreover, in this way we can apply techniques that involves effectivity to all separable metrizable spaces.
Indeed, given a second countable metrizable space X we can fix a compatible metric d and a dense subset $\boldsymbol{r}$, then for a suitable oracle $\varepsilon \in \omega^\omega$, the space $(X, \mathrm{d}, \boldsymbol{r})$ is an ε-recursively presented metric space. Thus the basis constituted by the open balls

$$V_n^{\mathcal{X}} = B(r((n)_0), q_{(n)_1}) = \{x \in X \mid \mathrm{d}(x, r((n)_0)) < q_{(n)_1}\}$$

makes it ε-recursive and, as remarked before, respect to this basis the relations R, S and T defined as

$$\begin{aligned}
S(m,n) &\Leftrightarrow \mathrm{d}(\boldsymbol{r}((m)_0)), \boldsymbol{r}((n)_0))) + q_{(m)_1} < q_{(n)_1} \\
R(m,n,p) &\Leftrightarrow S(p,n) \wedge S(p,m) \\
T(m,n,k) &\Leftrightarrow S(m,n) \wedge \mathrm{d}(\boldsymbol{r}((m)_0)), \boldsymbol{r}((k)_0))) > q_{(m)_1} + q_{(n)_1}
\end{aligned}$$

are ε-semirecursive. In this way we've defined on Y the structure $\mathcal{Y}$ of ε-recursive and ε-recursively regular space.

Remark 1.53. We observe that to build the structure of ε-recursive space for an arbitrary second countable metrizable space we have to use explicitly a metric and the ε-recursive representation. Thus, it becomes natural to consider the following question: is there another way to select explicitly an adequate countable basis of a second countable metrizable space X that makes it regular w.r.t. an oracle?

1.3.2 Good universal systems for arbitrary second coutable metrizable spaces

A second countable metrizable space Y, as showed in the previous section, has a structure of ε-recursive spaces $\mathcal{Y}$.
Given an oracle $\alpha \in \omega^\omega$ we denote with $W_e^{\varepsilon,\alpha}$ the e-th $\varepsilon \oplus \alpha$-semirecursive set of ω (that is $W_e^{\varepsilon,\alpha} = \{n \in \omega \mid \varphi_e^{\varepsilon\oplus\alpha}(n)\downarrow\}$). Similarly as we have defined the universal set in Theorem 1.40, we inductively define for each $\Sigma_n^{0,\varepsilon\oplus\alpha}$ a parametrization system $(H_{\Sigma_n^0}^{(\mathcal{Y},\varepsilon)})_{(\mathcal{Y},\varepsilon)}$:

$$\begin{aligned}
H_{\Sigma_1^0}^{(\mathcal{Y},\varepsilon)} &= \{(\alpha,e,y) \in \omega^\omega \times \omega \times Y \mid \exists n \in W_e^{\varepsilon,\alpha}(y \in V_n^{\mathcal{Y}})\} \\
H_{\Sigma_{n+1}^0}^{(\mathcal{Y},\varepsilon)} &= \{(\alpha,e,y) \in \omega^\omega \times \omega \times Y \mid \exists i \in \omega \neg H_{\Sigma_n^0}^{(\omega\times\mathcal{Y},\varepsilon)}(\alpha,e,i,y)\}
\end{aligned}$$

As for the case of Theorem 1.40, the α-section $H_{\Sigma_n^0,\alpha}^{(\mathcal{Y},\varepsilon)}$ parametrizes $\Sigma_n^{0,\varepsilon\oplus\alpha}(\mathcal{Y})$. Moreover, the set $G_{\Sigma_n^0}^{(\mathcal{Y},\varepsilon)} \subseteq \omega^\omega \times Y$ defined as:

$$G_{\Sigma_n^0}^{(\mathcal{Y},\varepsilon)}(e^\frown\alpha, y) \Leftrightarrow H_{\Sigma_n^0}^{(\mathcal{Y},\varepsilon)}(\alpha,e,y)$$

is $\Sigma_n^{0,\varepsilon}$ and universal for $\boldsymbol{\Sigma}_n^0(Y)$. We fix once and for all the two systems defined above and when ε or Y is understood from the context we will write only $G_{\Sigma_n^0}$ and $H_{\Sigma_n^0}$. For simplicity in the following, often we state results for $\mathcal{Y}$ recursive space, but all of them can be restated for $\mathcal{Y}$ ε-recursive space (and hence for second countable metrizable spaces with an adequate oracle).

Proposition 1.54. Given $\mathcal{Y}$ recursive space, then for every $n \geq 1$:

1. given $\mathcal{X}$ of type 0, exists an injective recursive function $S : \omega \times \mathcal{X} \to \omega$ such that:

$$\forall e \in \omega \forall x \in X \forall y \in Y (H^{\mathcal{X}\times\mathcal{Y}}_{\Sigma^0_n}(\alpha, e, x, y) \Leftrightarrow H^{\mathcal{Y}}_{\Sigma^0_n}(\alpha, S(e,x), y))$$

   In particular, the parametrization system $(H^{\mathcal{Y}}_{\Sigma^0_n})_{\mathcal{Y}}$ is *effectively good*.

2. given $\mathcal{X}$ of type 1, exists an injective continuous function $S : \omega^\omega \times \mathcal{X} \to \omega^\omega$ such that:

$$\forall \varepsilon \in \omega^\omega \forall x \in X \forall y \in Y (G^{\mathcal{X}\times\mathcal{Y}}_{\Sigma^0_n}(\varepsilon, x, y) \Leftrightarrow G^{\mathcal{Y}}_{\Sigma^0_n}(S(\varepsilon, x), y))$$

   In particular, the parametrization system $(G^{\mathcal{Y}}_{\Sigma^0_n})_{\mathcal{Y}}$ is *good*.

Proof. 1. Suppose that $\mathcal{X} = \omega^k$, then we have:

$$\begin{aligned} H^{\mathcal{X}\times\mathcal{Y}}_{\Sigma^0_1}(\alpha, e, x, y) &\Leftrightarrow (x_0, \dots, x_{k-1}, y) \in \bigcup_{n \in W^\alpha_e} V^{\mathcal{X}\times\mathcal{Y}}_n \\ &\Leftrightarrow (x_0, \dots, x_{k-1}, y) \in \bigcup_{n \in W^\alpha_e} \{(n)_0\} \times \cdots \times \{(n)_{k-1}\} \times V^{\mathcal{Y}}_{(n)_k} \\ &\Leftrightarrow \exists n \in W^\alpha_e (\bigwedge_{i<k} x_i = (n)_i \wedge y \in V^{\mathcal{Y}}_{(n)_k}) \end{aligned}$$

we now define:

$$h(e, x_0, \dots, x_n, m) = \begin{cases} 1 & \text{if } \exists n \in \omega (n \in W^\alpha_e \wedge \bigwedge_{i<k} x_i = (n)_i \wedge m = (n)_k) \\ \uparrow & \text{otherwise} \end{cases}$$

such function is α-computable because it graph is $\Sigma^{0,\alpha}_1$, hence $\varphi^\alpha_j = h$ for some $j \in \omega$ and by the S-m-n Theorem there is an injective function S such that:

$$\varphi^\alpha_j(e, x_0, \dots, x_n, m) = \varphi^\alpha_{S(j,e,x_0,\dots,x_n)}(m)$$

we observe that:

$$\begin{aligned} H^{\mathcal{Y}}_{\Sigma^0_1}(\alpha, S(j, e, x_0, \dots, x_n), y) &\Leftrightarrow y \in \bigcup_{m \in W^\alpha_{S(j,e,x_0,\dots,x_n)}} V^{\mathcal{Y}}_m \\ &\Leftrightarrow \exists n \in W^\alpha_e (\bigwedge_{i<k} x_i = (n)_i \wedge y \in V^{\mathcal{Y}}_{(n)_k}) \\ &\Leftrightarrow H^{\mathcal{X}\times\mathcal{Y}}_{\Sigma^0_1}(\alpha, e, x, y) \end{aligned}$$

The other equivalences follows by induction.

2. This case is similar to the previous one and for simplicity we only sketch the proof for $\mathcal{X} = \omega^k \times (\omega^\omega)^l$, observe that:

$$\begin{aligned}
G^{\mathcal{X}\times\mathcal{Y}}_{\Sigma^0_1}(e^\frown\varepsilon, \overline{x}, \overline{\gamma}, y) \Leftrightarrow & H^{\mathcal{X}\times\mathcal{Y}}_{\Sigma^0_1}(\varepsilon, e, \overline{x}, \overline{\gamma}, y) \\
\Leftrightarrow & H^{(\omega^\omega)^l\times\mathcal{Y}}_{\Sigma^0_1}(\varepsilon, S(e,\overline{x}), \overline{\gamma}, y) \\
\Leftrightarrow & (\overline{\gamma}, y) \in \bigcup_{n\in W^\varepsilon_{S(e,\overline{x})}} V^{(\omega^\omega)^l\times\mathcal{Y}}_n \\
\Leftrightarrow & (\overline{\gamma}, y) \in \bigcup_{n\in W^\varepsilon_{S(e,\overline{x})}} V^{\omega^\omega}_{(n)_0} \times\cdots\times V^{\omega^\omega}_{(n)_{l-1}} \times V^{\mathcal{Y}}_{(n)_l}
\end{aligned}$$

we recall that:

$$V^{\omega^\omega}_k = \prod_{i<\ell(s_k)} V^\omega_{s_k(i)} \times \prod_{i\geq\ell(s_k)} \omega = \prod_{i<\ell(s_k)} \{s_k(i)\} \times \prod_{i\geq\ell(s_k)} \omega$$

hence:

$$\begin{aligned}
& G^{\mathcal{X}\times\mathcal{Y}}_{\Sigma^0_1}(e^\frown\varepsilon, \overline{x}, \overline{\gamma}, y) \Leftrightarrow \\
& \qquad \exists m, n \in \omega (n \in W^\varepsilon_{S(e,\overline{x})} \wedge \bigwedge_{i<l} (s_{(n)_i} \prec \gamma_i) \wedge (n)_l = m \wedge y \in V^{\mathcal{Y}}_m)
\end{aligned}$$

from here we can proceed as before using the S-m-n Theorem on a $\varepsilon \oplus \gamma_0 \oplus \cdots \oplus \gamma_{l-1}$-computable function and complete the proof. □

Chapter 2

The Shore-Slaman Join Theorem in recursive spaces

Classically, Computability Theory defines a notion of computability for functions on the natural numbers $f : \omega \rightharpoonup \omega$. Then extend it to functions on finite strings $f : \omega^{<\omega} \rightharpoonup \omega^{<\omega}$ using an effective coding as the one we have defined in the introduction. However, the same approach cannot be used to induce a notion of computability on uncountable object as the Baire space ω^ω. Indeed, in this case, for cardinality reasons we cannot even use a bijection. In the previous chapter we have followed the approach of Effective Descriptive Set Theory to overcome this problem, but this is not the only way. In particular, there is the approach of Computable Analysis that starts from the concept of Type-2 theory of effectivity to extend the notion of computability to the Baire space ω^ω and then represent the points of topological spaces using elements in ω^ω. We see in the first section how these representations can be used to give a notion of computability and continuity between T_0 basic spaces that, in the end, coincide with the ones introduced in the previous chapter. In the last section we introduce the continuous degrees and present a proof of the Shore-Slaman Join Theorem.

2.1 Computability in T_0 basic spaces

2.1.1 Computability in the Baire space

Following [Val21] we now introduce a definition of computability on the Baire space that is equivalent to the one defined using a Type-2 Turing Machine. We recall that a Type-2 machine is like a standard Turing machine (with a read-only tape and finitely many work tapes), that it is allowed to run with an infinite string on the input tape. We say that a Type-2 machine *computes* a function $F : \omega^\omega \rightharpoonup \omega^\omega$ if, whenever executed with $x \in \text{dom}(F)$ on the input tape, it runs forever and, in the limit, writes $F(x)$ on the output tape (without mind changes). This concept can be equivalently expressed as:

Definition 2.1. A partial function $F : \omega^\omega \rightharpoonup \omega^\omega$ is **computable** if there is a computable function $\tilde{F} : \omega^{<\omega} \rightharpoonup \omega^{<\omega}$ such that:

1. $\tilde{F}$ is **$\preceq$-monotone**: $\forall s, t \in \text{dom}(\tilde{F})(s \preceq t \Rightarrow \tilde{F}(s) \preceq \tilde{F}(t))$
2. $\tilde{F}$ is an **approximating function** for F:

$$F(x) = y \Leftrightarrow \forall n \in \omega \exists m \geq n (y \upharpoonright n \preceq \tilde{F}(y) \upharpoonright m)$$

Observe that computable functions are closed under composition. Moreover, they are continuous on the Baire space, since continuous functions are exactly those that admits a monotone (not necessarily computable) approximating function. In particular, the following result holds:

Theorem 2.2 (Universal Turing Machine for continuous functions [Val21, Theorem 1.2])**.** There is a computable function $U : \omega^\omega \rightharpoonup \omega^\omega$ (called **universal computable function**) such that for every partial continuous function $f : \omega^\omega \rightharpoonup \omega^\omega$ there is a $p \in \omega^\omega$ such that $\forall x \in \text{dom}(f)(U(p \oplus x) = f(x))$.

In particular, it follows that partial continuous functions are exactly the computable ones with respect to some oracle.

Notice that, in general, even if a partial function $f : \omega^\omega \rightharpoonup \omega^\omega$ is continuous with respect to the subspace topology on $\text{dom}(f)$ it does not imply that such f can be extended to a continuous function defined on the entire space ω^ω.[1]

2.1.2 Continuity and computability in T_0 basic spaces

We now present the approach of Computable Analysis for characterizing the notion of Σ^0_1-recursivity and continuity. This approach lies on the idea of extending the notion of computability (and continuity) from the Baire space to T_0 basic spaces (and second countable T_0 topological spaces). The part regarding the continuity is quite classical and can be found in many sources (for example in [Wei00] or [Peq15]), while the part regarding the Σ^0_1-recursivity was never presented in these terms (although it is implicitly used in many articles — see for example [Val21] or [GKN21]).

[1]The same remark clearly holds for partial continuous functions between any topological spaces.

Admissible representations for secound countable T_0 basic spaces

Definition 2.3. Given X topological space and two functions $f, g : \omega^\omega \rightharpoonup X$ we define:

$$f \le_t g \Leftrightarrow \exists T : \omega^\omega \rightharpoonup \omega^\omega \text{ continuous } \forall p \in \mathrm{dom}(f)(f(p) = g \circ T(p))$$

In this case we say that f is **continuously reducible** to g.

Since the identity $\mathrm{id}_{\omega^\omega} : \omega^\omega \to \omega^\omega$ is continuous, the relation $\le_t$ define a quasi-orders, and from it we define the equivalence relation $\equiv_t$ in the usual way, i.e. $\equiv_t := \le_t \cap \ge_t$.

Definition 2.4. Given (X, τ) second countable T_0 topological space, a partial function $f : \omega^\omega \rightharpoonup X$ is an **admissible representation** (w.r.t. the topology τ) if it is continuous and is the $\le_t$-greatest element among the partial continuous function from ω^ω to X (i.e. for every $g : \omega^\omega \rightharpoonup X$ partial continuous $g \le_t f$).

It is easy to see that an admissible function $f : \omega^\omega \rightharpoonup X$ is surjective, since for every point $x \in X$ the constant function $c_x : \omega^\omega \to X$ is continuous. Fixed an admissible representation $\rho : \omega^\omega \rightharpoonup X$, an element $p \in \omega^\omega$ is a *name* (or ρ-name) for a point $x \in X$ if $\rho(p) = x$.

We now introduce an example of admissible representation for T_0 basic spaces. Actually, the function that we define is normally defined in the context of T_0 second countable topological spaces (see for example [Peq15]), but since we are interested also in comparing effectivity (in addition to continuity) we work directly in a framework meaningful for Σ^0_1-recursive functions.

Example 2.5 (An admissible representation for T_0 basic spaces)**.** Given a T_0 basic space $\mathcal{X} = (X, (V_n^{\mathcal{X}})_{n \in \omega})$ we define the function $\rho_{\mathcal{X}} : \omega^\omega \rightharpoonup \mathcal{X}$:

$$\rho_{\mathcal{X}}(p) = x \Leftrightarrow \mathrm{ran}(p) = \mathrm{N}_{\mathrm{base}}(x) := \{n \in \omega \mid x \in V_n^{\mathcal{X}}\}$$

That is we identify each point $x \in X$ with the elements of Baire that correspond to an enumeration of the neighborhood basis. Having (X, τ) T_0 ensure that $\rho_{\mathcal{X}}$ is a function on its domain.

The following proposition is a result from [Peq15], but stating it in the framework of T_0 basic spaces allows us to say that the function $\rho_{\mathcal{X}}$ is not only continuous but also Σ^0_1-recursive.

Proposition 2.6 ([Peq15, Theorem 1])**.** Given a T_0 basic space $\mathcal{X}$, then

1. $\rho_{\mathcal{X}} : \omega^\omega \rightharpoonup X$ is admissible

2. $\rho_{\mathcal{X}} : \omega^\omega \rightharpoonup X$ is open

3. $\rho_{\mathcal{X}} : \omega^\omega \rightharpoonup \mathcal{X}$ is Σ^0_1-recursive on its domain (and hence $\rho_{\mathcal{X}} : \omega^\omega \rightharpoonup X$ is continuous)

Proof. 1. We consider $f : \omega^\omega \rightharpoonup X$ continuous and fix $\pi : \omega \to \omega$ enumeration of ω where each natural number is repeated infinitely many times. We define by induction on the length of the strings $s \in \omega^{<\omega}$ the $\preceq$-monotone function $h^* : \omega^{<\omega} \to \omega^{<\omega}$ defining:

- $h^*(\varepsilon) = \varepsilon$ where ε is the empty string
- $h^*(s^\frown n) = \begin{cases} h^*(s)^\frown \pi(\ell(s)) & \text{if } f[N_{s^\frown n}] \subseteq V^{\mathcal{X}}_{\pi(\ell(s))} \\ h^*(s) & \text{otherwise} \end{cases}$

where $N_s = \{x \in \omega^\omega \mid s \prec x\}$. We define the partial continuous function $h : \omega^\omega \rightharpoonup \omega^\omega$ as $h(\beta) = \bigcup_{n\in\omega} h^*(\beta \upharpoonright n)$ and we claim that h witnesses that $f \leq_t \rho_{\mathcal{X}}$. That is $\forall \beta \in \mathrm{dom}(f)(f(\beta) = \rho_{\mathcal{X}} \circ h(\beta))$ or equivalently:

$$\forall \beta \in \mathrm{dom}(f) \forall n \in \omega (n \in ran(h(\beta)) \Leftrightarrow f(\beta) \in V^{\mathcal{X}}_n)$$

That holds because:

$\Leftarrow$ Having that $f(\beta) \in V^{\mathcal{X}}_n$, by continuity of f, implies that $\exists l \in \omega(f[N_{\beta\upharpoonright l+1}] \subseteq V^{\mathcal{X}}_n \wedge \pi(l) = n)$. Therefore, by construction $n \in \mathrm{ran}(h^*(\beta \upharpoonright l+1) \subseteq \mathrm{ran}(h(\beta)))$.

$\Rightarrow$ Suppose that $n \in \mathrm{ran}(h(\beta))$, then we consider the minimal $l \in \omega$ such that $n \in \mathrm{ran}(h^*(\beta \upharpoonright l+1))$. By definition of h^*, this means that $f[N_{\beta\upharpoonright l+1}] \subseteq V_{\pi(l)}$ and $\pi(l) = n$. Thus $f(\beta) \in V^{\mathcal{X}}_n$.

2. Given $N_s = \{x \in \omega^\omega \mid s \prec x\}$ for $s \in \omega^{<\omega}$, we have:

$$\rho_{\mathcal{X}}[N_s] = \bigcap_{k<\ell(s)} V^{\mathcal{X}}_{s_k}$$

is open because finite intersection of open sets and hence $\rho_{\mathcal{X}}$ is open.

3. We have $\forall \beta \in \mathrm{dom}(\rho_{\mathcal{X}})$

$$\begin{aligned} \rho_{\mathcal{X}}(\beta) \in V^{\mathcal{X}}_n &\Leftrightarrow \exists k(\beta(k) = n) \\ &\Leftrightarrow \exists m(\beta \in V^{\omega^\omega}_m \wedge \exists k \in \omega(\ell(s_{(m)_0}) > k \wedge s_{(m)_0}(k) = n) \\ &\qquad \wedge q_{(m)_1} \leq 2^{-\ell(s_{(m)_0})-1}) \end{aligned}$$

and the last expression is $\Sigma^0_1(\omega^\omega \times \omega)$. □

As we have already said, usually the representation $\rho_{\mathcal{X}} : \omega^\omega \rightharpoonup \mathcal{X}$ is defined for second countable T_0 spaces by fixing a countable basis. By repeating the previous proof one can see that such representation has exactly the same topological properties (i.e. it is continuous, admissible, and open).

Relative continuity and continuity in second countable T_0 spaces

Definition 2.7. Given X and Y second countable T_0 spaces, a function $f : X \rightharpoonup Y$ is **relatively continuous** if there are two admissible representations $\delta_X : \omega^\omega \rightharpoonup X$, $\delta_Y : \omega^\omega \rightharpoonup Y$ and a partial continuous function $F : \omega^\omega \rightharpoonup \omega^\omega$ (called **continuous realizer**) such that

$$\forall p \in \mathrm{dom}(f \circ \delta_X)(f \circ \delta_X(x) = \delta_Y \circ F(x))$$

$$\begin{array}{ccc} \omega^\omega & \xrightarrow{F} & \omega^\omega \\ {\scriptstyle \delta_X}\downarrow & & \downarrow{\scriptstyle \delta_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

By the maximality of admissible representations, it is easy to see that the previous definition doesn't depend on the choice of the representations. That is, $f : X \rightharpoonup Y$ admits a continuous realizer for some choice of admissible representations of X and Y if and only if it admits a continuous realiser for any choice of admissible representations.

We can compare the definition of relativity continuity with the usual definition of continuity. Actually for second countable T_0 topological spaces the two concepts coincide, and, as the proof show, this equivalence relies heavily on the admissibility of the functions (for this reason the notion of admissible representation is important).

Theorem 2.8 ([Peq15, Theorem 2])**.** Given X, Y second countable T_0 topological spaces, then:

$$f : X \rightharpoonup Y \text{ is relatively continuous } \Leftrightarrow f : X \rightharpoonup Y \text{ is continuous}$$

Proof.

- $\Leftarrow$ We fix two any admissible representations δ_X and δ_Y. Having $f : X \rightharpoonup Y$ continuous, then $f \circ \delta_X : \mathrm{dom}(\delta_X) \rightharpoonup Y$ is continuous. By admissibility of δ_Y, there exists a continuous $T : \omega^\omega \rightharpoonup \omega^\omega$ such that $f \circ \delta_X = \delta_Y \circ T$ on $\mathrm{dom}(f \circ \delta_X)$, hence f is relatively continuous.

- $\Rightarrow$ We fix an enumeration $(B^X_n)_{n \in \omega}$ of a countable basis for (X, τ_X). Consider the function $\rho_X : \omega^\omega \rightharpoonup X$ defined in a similar way as the one in Example 2.5, i.e.:

  $$\rho_X(p) = x \Leftrightarrow \mathrm{ran}(p) = \{n \in \omega \mid x \in B^X_n\}$$

  Such function is well defined because the space is T_0 and, as we have already observed, it is an open admissible representation. We can assume, without loss of generality, that the relative continuity of f is witnessed by ρ_X and any other admissible representation of Y, say δ_Y (we maintain such name to stress that it can be any of the admissible ones). Therefore, there exists a continuous function $F : \omega^\omega \rightharpoonup \omega^\omega$ such that:

$$\forall p \in \mathrm{dom}(f \circ \rho_X)(\delta_Y \circ F(p) = f \circ \rho_X(p))$$

Therefore $f \circ \rho_X : \omega^\omega \rightharpoonup Y$ is continuous. Therefore, given $U \in \mathbf{\Sigma}^0_1(Y)$ we have that for some $V \in \mathbf{\Sigma}^0_1(\omega^\omega)$:

$$(f \circ \rho_X)^{-1}[U] = \rho_X^{-1}[f^{-1}[U]] = V \cap \mathrm{dom}(f \circ \rho_X) = V \cap \rho_X^{-1}[\mathrm{dom}(f)]$$

and, since ρ_X is surjective, we have that:

$$f^{-1}[U] = \rho_X[V \cap \mathrm{dom}(f \circ \rho_X)] = \rho_X[V \cap \rho_X^{-1}[\mathrm{dom}(f)]] = \rho_X[V] \cap \mathrm{dom}(f)$$

and thus having that ρ_X is open implies that f is continuous on its domain. □

We notice that, this result in [Peq15] was stated for total continuous functions, but (as witnesses by our proof) it can be extended to partial continuous functions.

How to induce a meaningful notion of computability?

We now introduce an analog of continuous reducibility:

Definition 2.9. Given X topological space and two functions $f, g : \omega^\omega \rightharpoonup X$ we define:

$$f \leq_c g \Leftrightarrow \exists T : \omega^\omega \rightharpoonup \omega^\omega \text{ computable } \forall p \in \mathrm{dom}(f)(f(p) = g \circ T(p))$$

In this case we say that f is **computably reducible** to g.

Since the identity $\mathrm{id}_{\omega^\omega} : \omega^\omega \to \omega^\omega$ is computable, also the relation $\leq_c$ define a quasi-orders and induce an equivalence relation $\equiv_c$ (i.e. $\equiv_c := \leq_c \cap \geq_c$).

We would like to introduce an effective analog to admissibility, that is a version of admissibility for Σ^0_1-recursive functions w.r.t. the quasi-order $\leq_t$, this is desirable because it would give an equivalent definition of Σ^0_1-recursion in the framework of Computable Analysis. With this purpose in mind we consider the notion of computability on T_0 basic spaces induced by their admissible representations. That is:

Definition 2.10. Given $\mathcal{X}$ and $\mathcal{Y}$ T_0 basic spaces, then a function, a function $f : X \rightharpoonup Y$ is **$(\rho_\mathcal{X}, \rho_\mathcal{Y})$-computable** if there is a partial continuous function $F : \omega^\omega \rightharpoonup \omega^\omega$ (called **realizer**) such that

$$\forall p \in \mathrm{dom}(f \circ \rho_\mathcal{X})(f \circ \rho_\mathcal{X}(x) = \rho_\mathcal{Y} \circ F(x))$$

Is this definition of computability meaningful? We already know that, considering the Baire space as basic space, the function $\rho_{\omega^\omega} : \omega^\omega \rightharpoonup \omega^\omega$ is Σ^0_1-recursive (on its domain). Actually, we also have that it is computable (in the sense of Type-2 computability). Indeed, we have the following:

Lemma 2.11. The function $\rho_{\omega^\omega} : \omega^\omega \rightharpoonup \omega^\omega$ satisfies the following properties:

1. ρ_{ω^ω} is computable
2. ρ_{ω^ω} has a computable right-inverse $\rho^{-1}_{\omega^\omega,dx}$

Proof.

1. With respect to the prefix metric (see example 1.14), the N_s are exactly the open balls: $N_s = \{x \in \omega^\omega \mid s \prec x\} = B(s^\frown 0^\infty, 2^{-\ell(s)})$ and for any $r > 0$, taking $m = \max\{n \in \omega \mid r \leq 2^{-n-1}\}$ we get $B(x,r) = N_{x \upharpoonright m}$.
We now define $\preceq$-monotone computable function approximating ρ_{ω^ω}

```
def rho_approx(α ↾ k):

    l = max{n ∈ ω | q_{(α(0))_1} ≤ 2^{-n-1}}
    v = s_{(α(0))_0}⌢0^∞ ↾ l
    for i in range(1, k):
    #for cicle with i ranging in {1, ..., k-1}
        h = max{n ∈ ω | q_{(α(i))_1} ≤ 2^{-n-1}}
        w = s_{(α(i))_0}⌢0^∞ ↾ h
        if v ⪯ w:
            v = w
        else if w ≺ v:
            v = v
        else:
            ↑ #if two elements of the image of α are
               incompatible then ρ_{ω^ω} is undefined

    return v
```

2. We define a $\preceq$-monotone computable function approximating a right-inverse of ρ_{ω^ω}

```
def rho_inv_approx(x ↾ k):

    v = "" #v start as the empty string ε
    for j in range(0, k+1): #Thanks to the double for loop we
                             guarantee the ⪯-monotonicity
        for i in range(0, j):
            l = max{n ∈ ω | q_{(i)_1} ≤ 2^{-n-1}}
            if s_{(i)_0}⌢0^l ↾ l = x ↾ l and l < j:
```

```
            #The second condition is needed because we know
             as input only the first k digits of x
                v = v^i
    return v
```

We observe that this function $\rho^{-1}_{\omega^\omega,ap}$ is defined on $\omega^{<\omega}$ and is $\preceq$-monotone because given t proper extension of s, before adding elements on the output of the algorithm it adds also all the element given from the computation of s and then proceed considering also the new elements in the condition (in the if statement). We now prove that the induced function $\rho^{-1}_{\omega^\omega,dx} : \omega^\omega \rightharpoonup \omega^\omega$ associates to each element a name. That is:

$$\forall n \in \omega (x \in V^{\omega^\omega}_n \Rightarrow \exists m \in \omega (\rho^{-1}_{\omega^\omega,dx}(x)(m) = n))$$

which is equivalent to:

$$\forall n \in \omega (x \in V^{\omega^\omega}_n \Rightarrow \exists m, k \in \omega (\ell(\rho^{-1}_{\omega^\omega,ap}(x \upharpoonright k)) \geq m \wedge \rho^{-1}_{\omega^\omega,ap}(x \upharpoonright k)(m) = n))$$

and this is true because $x \in V^{\omega^\omega}_n = B(s_{(n)_0}, q_{(n)_1})$ implies that considered $l = \max\{m \in \omega \mid q_{(n)_1} \leq 2^{-m-1}\}$ then $s_{(n)_0}{}^\frown 0^l \upharpoonright l = x \upharpoonright l$. Therefore, considered any $k \geq l$ there is a suitable m (that can be found following the algorithm) and both verify the condition above. □

By the previous lemma, we have $\mathrm{id}_{\omega^\omega} \equiv_c \rho_{\omega^\omega}$ indeed:

$$\rho_{\omega^\omega} = \mathrm{id}_{\omega^\omega} \circ \rho_{\omega^\omega} \wedge \mathrm{id}_{\omega^\omega} = \rho_{\omega^\omega} \circ \rho^{-1}_{\omega^\omega,dx}$$

therefore, the notion of $(\rho_{\mathcal{X}}, \rho_{\mathcal{Y}})$-computability extends the notion of computability on the Baire space. Where by "extend" we mean that the following holds

Proposition 2.12. Given a function $f : \omega^\omega \rightharpoonup \omega^\omega$, then

$$f \text{ is computable} \Leftrightarrow f \text{ is } (\rho_{\omega^\omega}, \rho_{\omega^\omega})\text{-computable.}$$

Proof. $\Rightarrow$ Given $f : \omega^\omega \rightharpoonup \omega^\omega$ computable we consider as realizer $F = \rho^{-1}_{\omega^\omega,dx} \circ f \circ \rho_{\omega^\omega}$ (that is computable by the previous lemma). Hence, for any $p \in \mathrm{dom}(f \circ \rho_{\omega^\omega})$

$$f \circ \rho_{\omega^\omega}(p) = \underbrace{\rho_{\omega^\omega} \circ \rho^{-1}_{\omega^\omega,dx}}_{=\mathrm{id}_{\omega^\omega}} \circ f \circ \rho_{\omega^\omega}(p) = \rho_{\omega^\omega} \circ F(p)$$

$\Leftarrow$ On the other hand, f $(\rho_{\omega^\omega}, \rho_{\omega^\omega})$-computable means that here exists a computable F such that

$$\forall p \in \mathrm{dom}(f \circ \rho_{\omega^\omega})(\rho_{\omega^\omega} \circ F(p) = f \circ \rho_{\omega^\omega}(p))$$

therefore, since ρ_{ω^ω} is surjective, for any $x \in \mathrm{dom}(f)$ we have

$$f(x) = \rho_{\omega^\omega} \circ F(\rho^{-1}_{\omega^\omega, dx}(x)) = \rho_{\omega^\omega} \circ F \circ \rho^{-1}_{\omega^\omega, dx}(x)$$

Thus, thanks to the previous lemma, f is computable. □

Moreover, $(\rho_{\mathcal{X}}, \rho_Y)$-computability coincides with the notion of Σ^0_1-recursivity from Effective Descriptive Set Theory:

Proposition 2.13 ([Val21, Proposition 1.27])**.** Given $\mathcal{X}$ and $\mathcal{Y}$ T_0 basic spaces considered with their admissible representations $\rho_{\mathcal{X}}$ and $\rho_{\mathcal{Y}}$, we have

$f : X \rightharpoonup Y$ is $(\rho_{\mathcal{X}}, \rho_{\mathcal{Y}})$-computable $\Leftrightarrow$ $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ is Σ^0_1-recursive on its domain

Proof. $\Rightarrow$ Suppose that f is computable with computable realizer F. We define the following Σ^0_1 sets:

$$G := \{(m,k) \in \omega^2 \mid \exists \sigma, \tau \in \omega^{<\omega}(\tilde{F}(\sigma) = \tau \wedge \exists j < \ell(\tau)(\tau(j) = k) \wedge R^*(\sigma, m))\}$$
$$P := \bigcup_{(m,k)\in G} V^{\mathcal{X}}_m \times \{k\} \subseteq X \times \omega$$

where $\tilde{F} : \omega^{<\omega} \rightharpoonup \omega^{<\omega}$ is the computable approximating function of the realizer F and $R^* \in \Sigma^0_1(\omega^{<\omega} \times \omega)$ is the predicate witnessing that $\mathcal{X}$ is basic for finite intersections (see the observation after Definition 1.1). We now prove that $P \cap (\mathrm{dom}(f) \times \omega) = D_f$ where D_f is the diagram of f.

- $\subseteq$ Given $\sigma, \tau \in \omega^{<\omega}$ such that $\tilde{F}(\sigma) = \tau$ then we have:

  $$f\left[\bigcap_{i<\ell(\sigma)} V^{\mathcal{X}}_{\sigma(i)}\right] \subseteq \bigcap_{j<\ell(\tau)} V^{\mathcal{Y}}_{\tau(j)}$$

  Therefore, if $(x,k) \in P \cap (\mathrm{dom}(f) \times \omega)$ and this is witnessed by the pair of finite strings $\sigma, \tau \in \omega^{<\omega}$, then $x \in \bigcap_{i<\ell(\sigma)} V^{\mathcal{X}}_{\sigma(i)}$ and hence $f(x) \in \bigcap_{j<\ell(\tau)} V^{\mathcal{Y}}_{\tau(j)} \subseteq V^X_k$.
- $\supseteq$ Considered $(x,n) \in D_f$, then for every name p of x there is a $\sigma \prec p$ such that $\exists j \in \omega(\tilde{F}(\sigma)(j) = n)$. Moreover, since $x \in \bigcap_{i<\ell(\sigma)} V^{\mathcal{X}}_{\sigma(i)}$, there is $q \in \omega$ such that $(q,n) \in G$ and $x \in V^{\mathcal{X}}_q$, hence $(x,n) \in V^{\mathcal{X}}_q \times \{n\}$. Therefore, $(x,n) \in P$.

$\Leftarrow$ Suppose that f is recursive on its domain with witness $P \in \Sigma^0_1(\mathcal{X} \times \omega)$. Therefore, by Proposition 1.6, there is some $P^* \subseteq \omega^2$ semirecursive such that:

$$(x,n) \in P \Leftrightarrow \exists m \in \omega(x \in V^{\mathcal{X}}_m \wedge P^*(m,n))$$

We prove that f has as computable realizer the function approximated by the following $\tilde{F} : \omega^{<\omega} \rightharpoonup \omega^{<\omega}$:

```
def F̃(p ↾ k):

    v = "" #v start as the empty string ε
    for i in range(0, k): #Thanks to the triple for loop we
                            guarantee the ≤-monotonicity
        for j in range(0, i+1):
            for l in range(0, i+1):
                if P*(p(l), j):
                    v = v^j
    return v
```

$\tilde{F}$ is $\preceq$-monotone by construction. Therefore, we only have to prove that given p name of $x \in \mathrm{dom}(f)$, we have to prove that $F(p)$ is a name for $f(x)$. This is straightforward indeed:

$$\begin{aligned} f(x) \in V_n^{\mathcal{X}} &\Leftrightarrow (x,n) \in D_f \\ &\Leftrightarrow P(x,n) \Leftrightarrow \exists m \in \omega (x \in V_m^{\mathcal{X}} \wedge P^*(m,n)) \\ &\Leftrightarrow \exists l \in \omega (F(p)(l) = n) \end{aligned}$$ □

This suggest to give the following definition:

Definition 2.14. Given $\mathcal{X}$ T_0 basic space, a partial function $f : \omega^\omega \rightharpoonup \mathcal{X}$ is an **effectively-admissible representation** if it is Σ^0_1-recursive on its domain and is the $\leq_c$-greatest element among the partial Σ^0_1-recursive functions from ω^ω to $\mathcal{X}$ (i.e. for every $g : \omega^\omega \rightharpoonup \mathcal{X}$ Σ^0_1-recursive on its domain $g \leq_c f$).

Proposition 2.15. Given $\mathcal{X}$ T_0 basic space, the function $\rho_{\mathcal{X}} : \omega^\omega \rightharpoonup \mathcal{X}$ is an effectively-admissible representation.

Proof. The fact that $\rho_{\mathcal{X}}$ is Σ^0_1-recursive on its domain is proved in Proposition 2.6. Thus we only have to prove that for every $g : \omega^\omega \rightharpoonup \mathcal{X}$ Σ^0_1-recursive on its domain $g \leq_c \rho_{\mathcal{X}}$. Thanks to Proposition 2.13, the function g is $(\rho_{\omega^\omega}, \rho_{\mathcal{X}})$-computable and hence for some computable $G : \omega^\omega \rightharpoonup \omega^\omega$

$$\begin{array}{ccc} \omega^\omega & \xrightarrow{G} & \omega^\omega \\ {\scriptstyle \rho_{\omega^\omega}}\downarrow & & \downarrow{\scriptstyle \rho_{\mathcal{X}}} \\ \omega^\omega & \xrightarrow{g} & X \end{array}$$

Moreover, since ρ_{ω^ω} is surjective and has a computable right-inverse, $g = \rho_{\mathcal{X}} \circ G \circ \rho^{-1}_{\omega^\omega, dx}$. Therefore, $g \leq_c \rho_{\mathcal{X}}$ as wintessed by $T = G \circ \rho^{-1}_{\omega^\omega, dx}$. □

Notice that, since $\rho_{\mathcal{X}}$ is surjective, any effectively-admissible representation is surjective.

Definition 2.16. Given $\mathcal{X}$ and $\mathcal{Y}$ T_0 basic spaces, a function $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ is **relatively computable** if there are two effectively-admissible representations $\delta_{\mathcal{X}} : \omega^\omega \rightharpoonup \mathcal{X}$, $\delta_{\mathcal{Y}} : \omega^\omega \rightharpoonup \mathcal{Y}$ and a partial computable function $F : \omega^\omega \rightharpoonup \omega^\omega$ (called **computable realizer**) such that

$$\forall p \in \mathrm{dom}(f \circ \delta_{\mathcal{X}})(f \circ \delta_{\mathcal{X}}(x) = \delta_Y \circ F(x))$$

In analogy with the continuous case, we have that the previous definition does not depend on the choice of the representations. That is: $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ admits a computable realizer for some choice of effectively-admissible representations of $\mathcal{X}$ and $\mathcal{Y}$ if and only if it admits a computable realizer for any choice of effectively-admissible representations. In fact, holds:

Proposition 2.17. Given $\mathcal{X}$ and $\mathcal{Y}$ T_0 basic spaces, a function $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ is relatively-computable if and only if it is $(\rho_{\mathcal{X}}, \rho_{\mathcal{Y}})$-computable.

Proof. Suppose that the relative computability is witnessed by the effectively-admissible representations $\delta_{\mathcal{X}}$ and $\delta_{\mathcal{Y}}$.

$\Rightarrow$ We have

$$\forall p \in \mathrm{dom}(f \circ \delta_{\mathcal{X}})(\delta_{\mathcal{Y}} \circ F(p) = f \circ \delta_{\mathcal{X}}(p))$$

and since $\delta_{\mathcal{Y}} \leq_c \rho_{\mathcal{Y}}$, $\forall r \in \mathrm{dom}(\delta_{\mathcal{Y}})(\delta_{\mathcal{Y}}(r) = \rho_{\mathcal{Y}} \circ C_1(r))$ with C_1 computable, hence:

$$\forall p \in \mathrm{dom}(f \circ \delta_{\mathcal{X}})(\rho_{\mathcal{Y}} \circ C_1 \circ F(p) = f \circ \delta_{\mathcal{Y}}(p))$$

Similarly, $\rho_{\mathcal{X}} \leq_c \delta_{\mathcal{X}}$ that is $\forall q \in \mathrm{dom}(\rho_{\mathcal{X}})(\rho_{\mathcal{X}}(q) = \delta_{\mathcal{X}} \circ C_2(q))$ for a computable C_2 implies:

$$\forall q \in \mathrm{dom}(f \circ \rho_{\mathcal{X}}) \subseteq \mathrm{dom}(\rho_{\mathcal{X}})(\rho_{\mathcal{Y}} \circ \underbrace{C_1 \circ F \circ C_2}_{\text{computable}}(q) = f \circ \delta_{\mathcal{X}} \circ C_2(q) = f \circ \rho_{\mathcal{X}}(q))$$

$\Leftarrow$ Notice that in the previous direction we've only used that two effectively-admissible representations are (by definition) in the same equivalence class w.r.t. $\equiv_c$, thus this direction is proved in analogous way. □

Therefore, as for relative continuity, we have

Corollary 2.18. Given $\mathcal{X}$, $\mathcal{Y}$ T_0 basic spaces, then:

$f : \mathcal{X} \rightharpoonup \mathcal{Y}$ is relatively computable $\Leftrightarrow f : \mathcal{X} \rightharpoonup \mathcal{Y}$ is Σ^0_1-recursive on its domain

In light of this result, from now on we will not distinguish between the terms Σ^0_1-recursive and computable (we omit 'relatively' for simplicity) for functions between T_0 basic spaces.

2.1.3 Some representations

In general, in Computable Analysis with the term *representation* is intended any (partial) surjection from the Baire space. In this sense we present some examples of representations.

A representation for 0-dimensional recursive spaces

Given a T_0 basic space $\mathcal{X}$ (but also a second countable T_0 topological space), we have that any point is identified by its neighborhood basis $N_{\text{base}}(x)$. Moreover, we used this identification for building our (effectively-)admissible representation $\rho_{\mathcal{X}}$. A naive question might be: why do not use as representation the function that associate to any characteristic function of $N_{\text{base}}(x)$ the corresponding element x? A short answer is that such a function would not be admissible. Here we develop all the details and show a class of recursive spaces for which this representation is actually (effectively-)admissible.

Example 2.19 (An injective representation for T_0 basic spaces)**.** Given a T_0 basic space $\mathcal{X} = (X, (V_n^{\mathcal{X}})_{n\in\omega})$ we define the function $\delta_{\mathcal{X}} : \omega^\omega \rightharpoonup \mathcal{X}$:

$$\delta_{\mathcal{X}}(p) = x \Leftrightarrow \forall n \in \omega(p(n) = \chi_{N_{\text{base}}(x)}(n))$$

As we've already said, we identify each point $x \in X$ with the element of the Cantor space, that corresponds to the characteristic function of the neighborhood basis. As for $\rho_{\mathcal{X}}$, the fact that (X, τ) is T_0 ensure that $\delta_{\mathcal{X}}$ is a function on its domain. Moreover, it is injective and Σ^0_1-recursive on its domain, indeed $\forall p \in \text{dom}(\delta_{\mathcal{X}})$:

$$\delta_{\mathcal{X}}(p) \in V_n^{\omega^\omega} \Leftrightarrow p(n) = 1$$

Although this representation is injective and continuous and hence it seems more desiderable, it has the drawback of not being always admissible. In fact, using an argument in [Peq15, Proposition 5.3], we show:

Proposition 2.20. Given a T_0 basic space $\mathcal{X} = (X, (V_n)_{n\in\omega})$, we have that:

$$\delta_{\mathcal{X}} \text{ admissible} \Rightarrow X \text{ 0-dimensional}$$

Proof.

Claim 1. There is a $D \subseteq \text{dom}(\delta_{\mathcal{X}})$ such that $\delta_{\mathcal{X}} \upharpoonright D$ is still an admissible representation of $\mathcal{X}$ but it is also open.

Proof of the Claim. By admissibility $\rho_{\mathcal{X}} \leq_t \delta_{\mathcal{X}}$, so

$$\exists T : \omega^\omega \rightharpoonup \omega^\omega \text{ continuous } \forall p \in \text{dom}(\rho_{\mathcal{X}})(\rho_{\mathcal{X}}(p) = \delta_{\mathcal{X}} \circ T(p))$$

Moreover, without loss of generality, we can assume that $\mathrm{dom}(T) = \mathrm{dom}(\rho_{\mathcal{X}})$. The desired set is $D = T[\mathrm{dom}(\rho_{\mathcal{X}})] = \mathrm{ran}(T)$, indeed the same T witnesses that $\rho_{\mathcal{X}} \leq_t \delta_{\mathcal{X}} \upharpoonright D$, so in particular $\delta_{\mathcal{X}} \upharpoonright D$ is still admissible. For every $U \in \mathbf{\Sigma}^0_1(\omega^\omega)$ then $T^{-1}[U] = V \cap \mathrm{dom}(T)$ for some $V \in \mathbf{\Sigma}^0_1(\omega^\omega)$ and hence

$$\begin{aligned}\delta_{\mathcal{X}} \upharpoonright D[U] &= \{\delta_{\mathcal{X}} \circ T(\alpha) \mid \alpha \in T^{-1}[U]\} = \rho_{\mathcal{X}}[T^{-1}[U]] \\ &= \rho_{\mathcal{X}}[V \cap \mathrm{dom}(T)] = \rho_{\mathcal{X}}[V \cap \mathrm{dom}(\rho_{\mathcal{X}})] = \rho_{\mathcal{X}}[V] \in \mathbf{\Sigma}^0_1(\mathcal{X})\end{aligned}$$

thanks to the openness of $\rho_{\mathcal{X}}$. □

Moreover, $\delta_{\mathcal{X}} \upharpoonright D$ is surjective (because it is admissible) and injective, hence $D = \mathrm{dom}(\delta_{\mathcal{X}})$. Therefore, it is an homeomorphism from $\mathrm{dom}(\delta_{\mathcal{X}}) \subseteq \omega^\omega$ to X, and hence X is 0-dimensional. □

In general, $\delta_{\mathcal{X}}$ is not admissible (or effectively-admissible), however it is so for a precise class of recursive spaces (corresponding to the 0-dimensional ones).

Definition 2.21. A recursive space $\mathcal{X}$, is said **0-dimensional recursive** if the set $\{(x,n) \mid x \notin V^{\mathcal{X}}_n\} \in \Sigma^0_1(\mathcal{X} \times \omega)$.

Examples of 0-dimensional recursive spaces are ω, ω^ω and 2^ω. Notice that, for $\mathcal{X}$ 0-dimensional recursive, the inverse $\delta^{-1}_{\mathcal{X}} : \mathcal{X} \to \omega^\omega$ is Σ^0_1-recursive, indeed:

$$\delta^{-1}_{\mathcal{X}}(x) \in V^{\mathcal{X}}_n \Leftrightarrow \forall i \leq \ell(s_n)((s_n(i) = 0 \wedge x \notin V^{\mathcal{X}}_i) \vee (s_n(i) = 1 \wedge x \in V^{\mathcal{X}}_i))$$

Hence in this case $\delta_{\mathcal{X}} : \omega^\omega \rightharpoonup \mathcal{X}$ is open.

Proposition 2.22. Given $\mathcal{X}$ 0-dimensional recursive space, then $\rho_{\mathcal{X}} \leq_c \delta_{\mathcal{X}}$. In particular, $\delta_{\mathcal{X}} : \omega^\omega \rightharpoonup \mathcal{X}$ is admissible and effectively-admissible.

Proof. Observe that since $\mathcal{X}$ 0-dimensional recursive space exists some $C \in \Sigma^0_1(\omega^2)$ such that

$$x \notin V^{\mathcal{X}}_n \Leftrightarrow \exists m(x \in V^{\mathcal{X}}_m \wedge C(m,n))$$

We define the function $T : \omega^\omega \rightharpoonup \omega^\omega$ such that given $p \in \mathrm{dom}(\rho_{\mathcal{X}})$

$$T(p)(n) = \begin{cases} 1 & \text{if } \exists k(p(k) = n) \\ 0 & \text{if } \exists l \exists m(p(l) = m \wedge C(m,n)) \end{cases}$$

this function is computable and witnesses that $\rho_{\mathcal{X}} \leq_c \delta_{\mathcal{X}}$ indeed for any $p \in \mathrm{dom}(\rho_{\mathcal{X}})$ and $n \in \omega$

$$\rho_{\mathcal{X}}(p) \in V^{\mathcal{X}}_n \Leftrightarrow \exists k(p(k) = n) \Leftrightarrow T(p)(n) = 1$$

and

$$\rho_{\mathcal{X}}(p) \notin V_n^{\mathcal{X}} \Leftrightarrow \exists m(\rho_{\mathcal{X}}(p) \in V_m^{\mathcal{X}} \wedge C(m,n))$$
$$\Leftrightarrow \exists m, l(p(l) = m \wedge C(m,n)) \Leftrightarrow T(p) = 0$$

The admissibility follows from the maximality of $\rho_{\mathcal{X}}$. □

A representation for the Hilbert Cube

We recall the notion of effective compactness from [GKN21], that is useful to define a representation function widely used in their article. Remind that recursively presented metric spaces are recursive spaces (and recursively regular) if equipped with the basis given by the sets $V_n = B(r((n)_0), q_{(n)_1})$.

Definition 2.23. Given an oracle $\varepsilon \in \omega^\omega$, a recursively presented metric space $\mathcal{H}$ is ε**-effectively compact** if it is compact and there is an ε-computable function $\tilde{k} : \omega \to \omega$ deciding whether any finite collection of sets in $(V_n^{\mathcal{X}})_{n \in \omega}$ covers $\mathcal{H}$.[2] In particular, if $\varepsilon = \varnothing$ we say that $\mathcal{H}$ is **effectively compact**.

Recall that the Hilbert cube $[0,1]^\omega$ is an effectively compact space. Moreover, every compact Polish space is ε-effectively compact w.r.t. some oracle $\varepsilon \in \omega^\omega$. We now prove a property for ε-effectively compact spaces that is stated without proof in [GKN21, Section 2.2].

Proposition 2.24. Given an ε-recursively presented metric space $\mathcal{H}$ that is ε-effectively compact then $\{\langle d, e\rangle \in \omega \mid H \smallsetminus G_{\Sigma^0_1, d^\frown\varepsilon} \subseteq G_{\Sigma^0_1, e^\frown\varepsilon}\} \in \Sigma^{0,\varepsilon}_1(\omega)$ where $G_{\Sigma^0_1, e^\frown\varepsilon} = G^{\mathcal{H}}_{\Sigma^0_1, e^\frown\varepsilon}$ (and $(G^{\mathcal{Y}}_{\Sigma^0_1})_{\mathcal{Y}}$ is the universal system for $\boldsymbol{\Sigma}^0_1$).

Proof. Given $d, e \in \omega$, consider the open sets $G_{\Sigma^0_1, d^\frown\varepsilon}$ and $G_{\Sigma^0_1, e^\frown\varepsilon}$ we have that:

$$H \smallsetminus G_{\Sigma^0_1, d^\frown\varepsilon} \subseteq G_{\Sigma^0_1, e^\frown\varepsilon} \Leftrightarrow \{G_{\Sigma^0_1, d^\frown\varepsilon}, G_{\Sigma^0_1, e^\frown\varepsilon}\} \text{ is a covering of } \mathcal{X}$$

In particular, the last condition is equivalent to require that $\{V_m^{\mathcal{H}} \mid m \in W_e^\varepsilon \cup W_d^\varepsilon\}$ is a covering where W_e^ε and W_d^ε are the corresponding $\Sigma^{0,\varepsilon}_1$ sets in ω. Therefore, to positively decide if $H \smallsetminus G_{\Sigma^0_1, d^\frown\varepsilon} \subseteq G_{\Sigma^0_1, e^\frown\varepsilon}$ we can enumerate the elements in $W_e^\varepsilon \cup W_d^\varepsilon$ and then check (after every new element) with the function witnessing the effective compactness $\tilde{k}$ whether the set enumerated so far is a finite covering. □

[2] To be precise, the argument of the function h is a number coding a finite set of indices (of the sets in the basis). To see how to code a finite set we refer the reader to [Ter04, Section 2.4.4].

Example 2.25 (A representation for the Hilbert cube[3] [GKN21, Example 2.2])**.** We consider for each $n \in \omega$ the family $\mathcal{F}_n$ of open balls of diameter smaller than 2^{-n-1}. By effective compactness, we can extract a finite cover from each $\mathcal{F}_n$ that we call $\mathcal{C}_n = \{B^n_m \mid m \leq h(n)\}$ where $h : \omega \to \omega$ is computable and each $\mathcal{C}_{n+1}$ refines $\mathcal{C}_n$. Moreover, we have that $\{(l, k, m, n) \in \omega^4 \mid \hat{B}^l_k \subseteq B^n_m\}$ (where $\hat{B}^l_k$ is the closure of the ball B^l_k) is computable, indeed l correspond to the radius 2^{-l-2} and k to the center $r_{s(l,k)}$ where $s : \omega \times \omega \to \omega$ (i.e. the function that returns the index of the center of the l-th element of $\mathcal{C}_k$) is computable, and hence

$$\hat{B}^l_k \subseteq B^n_m \Leftrightarrow d(r_{s(l,k)}, r_{s(n,m)}) + 2^{-l-2} < 2^{-m-2}$$

is computable (because $[0,1]^\omega$ is recursively present). In addition, without loss of generality we can modify $h : \omega \to \omega$ in such a way that $\forall n \in \omega \forall k \in \omega(k < h(n) \Rightarrow \exists l \in \omega(l < h(n+1) \wedge \hat{B}^{n+1}_l \subseteq B^n_k))$.
We define the **tree $\boldsymbol{H}$ of names of elements in** $[0,1]^\omega$ by recursion in the following way:

- the empty string $\varepsilon \in \boldsymbol{H}$
- the unary strings $n \in \boldsymbol{H}$ if $n < h(0)$
- for each $\sigma^\frown k \in \boldsymbol{H}$ with $\ell(\sigma^\frown k) = l + 1$ then $\sigma^\frown k^\frown n \in \boldsymbol{H}$ if $n < h(l+1)$ and $\hat{B}^{l+1}_n \subseteq B^l_k$.

By construction, $\boldsymbol{H}$ is a recursively bounded recursive tree with no terminal nodes (hence $\varnothing \neq [\boldsymbol{H}] \in \Pi^0_1(\omega^\omega)$). For each $\sigma \in \boldsymbol{H}$ we define $B^*_\sigma \coloneqq B^{\ell(\sigma)-1}_{\sigma(\ell(\sigma)-1)}$ (we have $\mathrm{diam}(B^*_\sigma) < 2^{-\ell(\sigma)}$). Therefore, the map $\kappa : [\boldsymbol{H}] \to [0,1]^\omega$ defined as $\kappa(\beta) \in \bigcap_{n\in\omega} B^*_{\beta\restriction n} \in [0,1]^\omega$ is well defined and is a surjection.

Proposition 2.26. $\kappa : [\boldsymbol{H}] \to [0,1]^\omega$ is Σ^0_1-recursive (and hence continuous).

Proof. We have that the diagram of κ is:

$$\kappa(\beta) \in V_n = B(r((n)_0), q_{(n)_1}) \Leftrightarrow \exists l \in \omega(\beta \restriction l \in \boldsymbol{H} \wedge \hat{B}^*_{\beta\restriction l} \subseteq B(r((n)_0), q_{(n)_1}))$$

similarly as above $\hat{B}^{l-1}_{\beta(l-1)} \subseteq B(r((n)_0), q_{(n)_1})$ is a recursive predicate about $(l-1, \beta(l-1), n)$. □

In particular, being $\kappa : [\boldsymbol{H}] \to [0,1]^\omega$ Σ^0_1-recursive, we have $\kappa \leq_c \rho_{[0,1]^\omega}$.

[3] The same construction can be done for any effectively compact metric space, but we present it only for $[0,1]^\omega$ because we only use this one.

2.2 Continuous degrees

We recall from section 1.1.1 that, given two basic spaces $\mathcal{X}$ and $\mathcal{Y}$, $\Phi_e^{\mathcal{X},\mathcal{Y}}$ denotes the e-th partial Σ_1^0-recursive function (on its domain) from $\mathcal{X}$ to $\mathcal{Y}$ (that is the largest function induced by the e-th Σ_1^0 subset of ω^2).

Definition 2.27. Given $\mathcal{X}$ and $\mathcal{Y}$ basic spaces, $y \in Y$ is **representation reducible** to $x \in X$ (and we write $y \leq_M x$) if there is some $e \in \omega$ such that $\Phi_e^{\mathcal{X},\mathcal{Y}}(x) = y$.

If $\mathcal{X}$ and $\mathcal{Y}$ are recursive (and hence also T_0) then, thanks to Proposition 2.13, this definition coincides with the one given for recursively presented metric spaces in [GKN21, Definition 2.3].

Moreover, for recursive spaces this definition is equivalent to the one given in [Mos09, Exercise 3D.19] that is $y \leq_M x$ if and only if y is $\Sigma_1^0(x)$-recursive, i.e. $\mathrm{N}_{\mathrm{base}}(y) = \{n \in \omega \mid y \in V_n^{\mathcal{Y}}\} \in \Sigma_1^{0,x}$.

Lemma 2.28. Given $\mathcal{X}$ and $\mathcal{Y}$ recursive space, $x \in X$ and $y \in Y$ then

$$y \leq_M x \Leftrightarrow y \text{ is } \Sigma_1^0(x)\text{-recursive}$$

Proof. Recall that, accordingly to the definition in [Mos09, Section 3D] $\mathrm{N}_{\mathrm{base}}(y) \in \Sigma_1^{0,x}(\omega)$ means that there is a set $N \in \Sigma_1^0(\mathcal{X} \times \omega)$ such that

$$n \in \mathrm{N}_{\mathrm{base}}(y) \Leftrightarrow (x, n) \in N$$

- $\Rightarrow$ Suppose that $y = \Phi_e^{\mathcal{X},\mathcal{Y}}(x)$, therefore:

  $$\forall z \in \mathrm{dom}(\Phi_e^{\mathcal{X},\mathcal{Y}})(\Phi_e^{\mathcal{X},\mathcal{Y}}(z) \in V_n^{\mathcal{Y}} \Leftrightarrow P(z,n)) \quad \text{with } P \in \Sigma_1^0(\mathcal{X} \times \omega)$$

  In particular, for the given x we have that the x-section $P_x = \{n \mid \Phi_e^{\mathcal{X},\mathcal{Y}}(x) \in V_n^{\mathcal{Y}}\} \in \Sigma_1^0(\omega)$ and hence y is $\Sigma_1^0(x)$-recursive.

- $\Leftarrow$ On the other hand, y is $\Sigma_1^0(x)$-recursive means that there is an $N \in \Sigma_1^0(\mathcal{X} \times \omega)$ such that $\mathrm{N}_{\mathrm{base}}(y) = N_x \in \Sigma_1^0(\omega)$ thus we can consider the Σ_1^0-recursive function induced by N (more precisely, by the corresponding Σ_1^0 set in ω^2). Such function, by our assumptions, maps x into y and hence $y \leq_M x$. □

It is easy to see that $\leq_M$ is a quasi-order, indeed it is transitive because Σ_1^0-recursive functions are closed under composition, and reflexive since the identity is Σ_1^0-recursive. Thus, we have the equivalence relation $\equiv_M := \leq_M \cap \geq_M$ and we can define a degree structure:

Definition 2.29 (Continuous degrees)**.** Given $\mathcal{X}$ recursive space, the **continuous degree** of $x \in \mathcal{X}$ is its equivalence class under the relation $\equiv_M$ (over elements of recursive spaces). A point $x \in X$ is **total** if it is representation equivalent to an element $z \in 2^\omega$ (that is $x \equiv_M z$).

The name for this degree structure comes from the paper of Joseph S. Miller [Mil04] where they were defined and in which he proves that every continuous degree contains an element of $\mathcal{C}([0,1])$ (considered as recursively presented Polish space). However, for our purposes we need that:

Proposition 2.30. Every continuous degree contains an element of $[0,1]^\omega$.

Proof. This is basically a reformulation of Corollary 1.27. Indeed, it proves (a posteriori) that every recursive space $\mathcal{X}$ is recursively isomorphic to a subspace of the Hilbert cube. Thus $\forall x \in X \exists y \in [0,1]^\omega (x \equiv_M y)$. □

We recall that:

Definition 2.31. A function $\Psi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is an **enumerator operator** if there exists a set $P \in \Sigma^0_1(\omega \times \omega^{<\omega})$ such that:

$$\forall A \subseteq \omega(\forall k \in \omega(k \in \Psi(A) \Leftrightarrow \exists u \in \omega^{<\omega}(P(k,u) \wedge \forall i < \ell(u)(u(i) \in A))))$$

Definition 2.32. Given $A, B \in \mathcal{P}(\omega)$ we say that A is **enumeration reducible** to B ($A \leq_e B$) if there exists an enumerator operator $\Psi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that $\Psi(B) = A$.

We also recall other kinds of reducibility normally used in computability (on 2^ω and ω^ω):

- the *Turing reducibility* denoted by $\leq_T$, that is $\alpha \leq_T \beta$ if α is β-computable (i.e. $\alpha = \varphi^\beta_e$ for some $e \in \omega$).
- the *many-one reducibility* denoted by $\leq_m$, that is $\alpha \leq_m \beta$ if exists a total computable $f : \omega \to \omega$ such that $\forall n \in \omega(\alpha(n) = \beta(f(n)))$
- the *one-one reducibility* denoted by $\leq_1$, that is $\alpha \leq_1 \beta$ if exists a total computable injective $f : \omega \to \omega$ such that $\forall n \in \omega(\alpha(n) = \beta(f(n)))$

The corresponding equivalence relations are denoted as usual by: $\equiv_T$, $\equiv_m$ and $\equiv_1$. We now present a proof of [GKN21, Fact 2.4] for recursive spaces not present in the original paper.

Fact 1. Given $\mathcal{X}$ and $\mathcal{Y}$ recursive spaces, we have

1. $\forall x \in X \forall y \in Y (y \leq_M x \Leftrightarrow \mathrm{N}_{\mathrm{base}}(y) \leq_e \mathrm{N}_{\mathrm{base}}(x))$
2. $\forall z \in \omega^\omega (\mathrm{N}_{\mathrm{base}}(z) \equiv_m \{\sigma \in \omega^{<\omega} \mid \sigma \prec z\})$
3. $\forall x, y \in \omega^\omega (y \leq_T x \Leftrightarrow y \leq_M x)$
4. $\forall x \in X \forall z \in \omega^\omega (x \leq_M z \Leftrightarrow \exists p \in \omega^\omega (\rho_{\mathcal{X}}(p) = x \wedge p \leq_T z))$

5. $\forall x \in X(x \text{ is total } \Leftrightarrow \exists p \in \omega^\omega(\rho_\mathcal{X}(p) = x \wedge p \equiv_M x))$.
   In this case such p is called the **canonical name** of x.

Proof.

1. We prove something more precise: given $\mathcal{X}$ and $\mathcal{Y}$ recursive spaces and $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ then

   $$f\ \Sigma^0_1\text{–recursive on its domain} \Leftrightarrow \forall x \in \mathrm{dom}(f)(\mathrm{N}_{\mathrm{base}}(f(x)) \leq_e \mathrm{N}_{\mathrm{base}}(x))$$

   indeed, $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ is Σ^0_1-recursive on its domain means that $\forall x \in \mathrm{dom}(f)$:

   $$\forall n \in \omega(f(x) \in V^\mathcal{Y}_n \Leftrightarrow \exists m \in \omega(x \in V^\mathcal{X}_m \wedge W_e(n, m)))$$

   for some $e \in \omega$. That is:

   $$\forall n \in \omega(n \in \mathrm{N}_{\mathrm{base}}(f(x)) \Leftrightarrow \exists m \in \omega(W_e(n, m) \wedge m \in \mathrm{N}_{\mathrm{base}}(x)))$$

   Therefore, $P = W_e$ corresponds to the enumerator operator that witnesses $\mathrm{N}_{\mathrm{base}}(f(x)) \leq_e \mathrm{N}_{\mathrm{base}}(x)$.

2. We observe that $\mathrm{N}_{\mathrm{base}}(x) \leq_m \{\sigma \in \omega^{<\omega} \mid \sigma \prec x\}$ because:

   $$\begin{aligned} n \in \mathrm{N}_{\mathrm{base}}(x) \Leftrightarrow x \in V^{\omega^\omega}_n &\Leftrightarrow \mathrm{d}(x, r((n)_0)) < q_{(n)_1} \\ &\Leftrightarrow s_{(n)_0}{}^\frown 0^\infty \restriction \max\{k \in \omega \mid q_{(n)_1} \leq 2^{-k-1}\} \prec x \end{aligned}$$

   and similarly, using the same correspondence between balls (respect to the prefix metric) and prefixes of the elements in the Baire space one can prove the other direction.

3. Notice that considering Type-2 computability, given $x, y \in \omega^\omega$, $x \leq_T y$ if and only if there exists a computable $f : \omega^\omega \rightharpoonup \omega^\omega$ such that $y = f(x)$. Therefore, thanks to propositions 2.12 and 2.13, $x \leq_T y$ if and only if $x \leq_M y$.

4. Fix $x \in \mathcal{X}$ and $z \in \omega^\omega$, we have to prove that:

   $$x \leq_M z \Leftrightarrow \exists p \in \omega^\omega(\rho_\mathcal{X}(p) = x \wedge p \leq_T z)$$

   $\Leftarrow$: Having $\rho_\mathcal{X}(p) = x$ we have that $x \leq_M p$ (because this representation is Σ^0_1-recursive on its domain) and hence we conclude by the transitivity of $\leq_M$.

   $\Rightarrow$: We consider the Σ^0_1-recursive function on its domain $f : \omega^\omega \rightharpoonup \mathcal{X}$ which witnesses $f(z) = x$. By Proposition 2.13, we have that:

   $$\forall q \in \mathrm{dom}(\rho_{\omega^\omega})(\rho_{\omega^\omega}(q) = z \Rightarrow \rho_\mathcal{X} \circ F(q) = f \circ \rho_{\omega^\omega}(q) = f(z) = x)$$

Therefore, to conclude we have to select a name for x in the set $\{F(q) \mid q \text{ name of } z\}$, to do so we consider the right inverse of ρ_{ω^ω} given by Lemma 2.11, and take $\rho^{-1}_{\omega^\omega, dx}(z) = \tilde{p}$. Thus, $z \geq_M \tilde{p} \geq_M F(\tilde{p})$ and such $F(\tilde{p})$ is the desired name.

5. x total means that $\exists t \in 2^\omega (x \equiv_M t)$, therefore applying the previous point to $x \leq_M t$ we get:

$$\exists p \in \omega^\omega (\rho_X(p) = x \wedge p \leq_M t \equiv_M x)$$

and hence $p \equiv_M x$. □

We now recall a result on enumeration degrees that allows us to characterize the representation reducibility:

Lemma 2.33 (Selman, Rozinas [Coo04, Theorem 11.1.13])**.** Given $A, B \subseteq \omega$ we have

- $A \leq_e B \Leftrightarrow \forall C \subseteq \omega (\chi_B \in \Sigma^{0,\chi_C}_1(\omega) \Rightarrow \chi_A \in \Sigma^{0,\chi_C}_1(\omega))$
- $A \leq_e B \Leftrightarrow \forall C \subseteq \omega (B \leq_e C \oplus (\omega \smallsetminus C) \Rightarrow A \leq_e C \oplus (\omega \smallsetminus C))$

Notice that, as $\forall D, C (\chi_D \in \Sigma^{0,\chi_C}_1(\omega) \Leftrightarrow D \leq_e C \oplus (\omega \smallsetminus C))$, the points in the previous lemma are equivalent. In particular, this allows us to prove the following:

Corollary 2.34. Given $\mathcal{X}$ and $\mathcal{Y}$ recursive spaces:

$$\forall x \in X \forall y \in Y (x \leq_M y \Leftrightarrow \forall z \in 2^\omega (y \leq_M z \Rightarrow x \leq_M z))$$

Proof. $\Rightarrow$: This is immediate by the transitivity of $\leq_M$.

$\Leftarrow$: We prove the contrapositive. Suppose that $x \nleq_M y$, then $\mathrm{N}_{\text{base}}(x) \nleq_e \mathrm{N}_{\text{base}}(y)$. Therefore by the previous lemma there exists a $C \subseteq \omega$ such that $\mathrm{N}_{\text{base}}(y) \leq_e C \oplus (\omega \smallsetminus C)$ and $\mathrm{N}_{\text{base}}(x) \nleq_e C \oplus (\omega \smallsetminus C)$. Said $\chi_C : \omega \to 2$ the characteristic function of C, we have that $C \oplus (\omega \smallsetminus C) \equiv_e \mathrm{N}_{\text{base}}(\chi_C)$. Therefore, we get $\exists c \in 2^\omega (x \nleq_M c \wedge y \leq_M c)$. □

Another important property of continuous degrees is *almost totality*. Actually, such property characterize continuous degrees with respect to the enumeration degrees structure, indeed in [And+19] they found that any enumeration degree is almost total if and only if it is continuous.

Lemma 2.35 (Almost totality [And+19, Lemma 3.2], [GKN21, Lemma 2.6])**.** Given $\mathcal{X}$ recursive space, $\forall x \in X \forall z \in 2^\omega (z <_M x \dot{\vee} x \oplus z \text{ total})$.[4]

The results above allow us to reduce arguments involving continuous degrees and $\leq_M$ into arguments about Turing degrees on 2^ω (i.e. the total degrees).

[4]Where $\dot{\vee}$ denotes the exclusive disjunction.

2.2.1 The Turing Jump operator in recursive spaces

The notion of Turing Jump of a point on 2^ω (or on ω^ω) coincide with the halting problem relativized to such point that is:

$$\begin{aligned} J : 2^\omega &\to 2^\omega \\ x &\mapsto J(x) = \{e \in \omega \mid \varphi_e^x(e)\downarrow\} \end{aligned}$$

It can be extended to all recursive spaces and to all Σ_n^0 pointclasses as:

Definition 2.36. Given $\mathcal{X}$ recursive space and $n \geq 1$, the $\Sigma_n^{0,\alpha}$**-jump** is the function defined as:

$$\begin{aligned} J_{\mathcal{X}}^{(n),\alpha} : \mathcal{X} &\to 2^\omega \\ x &\mapsto J_{\mathcal{X}}^{(n),\alpha}(x) = \{e \in \omega \mid x \in H^{\mathcal{X}}_{\Sigma_n^0,\alpha,e}\} \end{aligned}$$

Where $(H^{\mathcal{Y}}_{\Sigma_n^0})_{\mathcal{Y}}$ is the universal system for Σ_n^0 introduced in the last section of Chapter 1.

Similarly, we define the $\Sigma_n^{0,\alpha}$-jump for ε-recursive spaces for $\alpha \geq_T \varepsilon$.
We observe that such definition extend the definition of Turing jump on the usual spaces (i.e. 2^ω and ω^ω). However, to prove this result we need the following property:

Proposition 2.37 (Finite use property [Ter04, Proposition 5.1.4])**.** Given $x \in 2^\omega$, suppose that $\varphi_n^x(n)[t]\downarrow$, then there is a finite string $s_x \preceq x$ such that $\varphi_n^{s_x}(n)[t]\downarrow$. In particular: $\forall y \in 2^\omega (s_x \preceq y \Rightarrow \varphi_n^y(n)[t]\downarrow)$.

Proposition 2.38. Said $J^{(1),\varnothing}_{\omega^\omega} : \omega^\omega \to 2^\omega$ the unrelativized[5] jump operator on the Baire space and $J : \omega^\omega \to 2^\omega$ the usual Turing jump, then

$$\forall x \in \omega^\omega (J^{(1),\varnothing}_{\omega^\omega}(x) \equiv_1 J(x))$$

Proof. Notice that the unrelativized jump operator on the Baire space is defined on any point $x \in \omega^\omega$ as the set:

$$J^{(1),\varnothing}_{\omega^\omega}(x) = \left\{ e \in \omega \,\middle|\, x \in \bigcup_{m \in W_e} V_m^{\omega^\omega} \right\}$$

clearly, $J^{(1),\varnothing}_{\omega^\omega}(x) \in \Sigma_1^{0,x}(\omega)$ and hence $J^{(1),\varnothing}_{\omega^\omega}(x) \leq_1 J(x)$ (where $\leq_1$ indicates the one-one reduction) [MP22, Corollary 5.4 Chapter 5][6]. For the other

[5]Here we identify the empty set $\varnothing$ with the sequence $0^\infty = 000\ldots$.

[6]Actually, this result in [MP22] is stated for many-one reduction $\leq_m$, but using an injective version of the S-m-n Theorem, it can be restated for $\leq_1$.

inequality we observe that $J(x) \in \Sigma_1^{0,x}$, indeed:

$$\begin{aligned} J(x) &= \{e \in \omega \mid \varphi_e^x(e)\downarrow\} \overset{\text{Finite use property}}{=} \{e \in \omega \mid \exists \sigma \in \omega^{<\omega}(\varphi_e^\sigma(e)\downarrow \wedge \sigma \prec x)\} \\ &= \{e \in \omega \mid \exists n \in \omega(\varphi_e^{s(n)_0}(e)\downarrow \wedge x \in V_n^{\omega^\omega})\} \end{aligned}$$

therefore, for some fixed $i \in \omega$

$$e \in J(x) \Leftrightarrow H_{\Sigma_1^0}^{\omega \times \mathcal{X}}(\varnothing, i, e, x) \Leftrightarrow H_{\Sigma_1^0}^{\mathcal{X}}(\varnothing, S(i,e), x) \Leftrightarrow S(i,e) \in J_{\omega^\omega}^{(1),\varnothing}(x)$$

where $S : \omega^2 \to \omega$ is the injective function witnessing that the system $(H_{\Sigma_n^0}^{\mathcal{Y}})_{\mathcal{Y}}$ is effectively good, thus $J(x) \leq_1 J_{\omega^\omega}^{(1),\varnothing}(x)$. □

Therefore, the usual Turing Jump on the Baire space is "equivalent" to the Σ_1^0-jump $J_{\omega^\omega}^{(1),\varnothing}$. The following lemmas confirm that the new jump operator behaves well with respect to the degree structure of continuous degrees. These results where stated in [GKN21] for recursively presented metric spaces, we show that the same proofs can be carried out in the framework of recursive spaces.

Lemma 2.39 ([GKN21, Lemma 2.8]). Given $\mathcal{X}$ recursive space, and $n, m \geq 1$, then:

1. $\forall x \in X \forall \alpha \in \omega^\omega (J_{\mathcal{X}}^{(n),\alpha}(x) \equiv_1 J_{\omega^\omega \times \mathcal{X}}^{(n),\varnothing}(\alpha, x))$
2. $\forall x \in X (J_{\mathcal{X}}^{(n+m),\varnothing}(x) \equiv_1 J^{(m)} \circ J_{\mathcal{X}}^{(n),\varnothing}(x))$

Proof. Both statements are a consequence of the fact that the system $(H_{\Sigma_n^0}^{\mathcal{Y}})_{\mathcal{Y}}$ is effectively good and the witnessing computable function S (of effective goodness) can be chosen to be injective (see Proposition 1.54).

1. Fixed $x \in \mathcal{X}$ and $\alpha \in \omega^\omega$ then $J_{\mathcal{X}}^{(n),\alpha}(x) \leq_1 J_{\omega^\omega \times \mathcal{X}}^{(n),\varnothing}(\alpha, x)$ because:

   $$\begin{aligned} e \in J_{\mathcal{X}}^{(n),\alpha}(x) \Leftrightarrow H_{\Sigma_n^0}^{\mathcal{X}}(\alpha, e, x) &\overset{H_{\Sigma_n^0}^{\mathcal{X}} \in \Sigma_n^0(\omega^\omega \times \omega \times \mathcal{X})}{\Leftrightarrow} H_{\Sigma_n^0}^{\omega^\omega \times \omega \times \mathcal{X}}(\varnothing, f, \alpha, e, x) \\ &\Leftrightarrow H_{\Sigma_n^0}^{\omega^\omega \times \mathcal{X}}(\varnothing, S(f,e), \alpha, x) \Leftrightarrow S(f,e) \in J_{\omega^\omega \times \mathcal{X}}^{(n),\varnothing}(\alpha, x) \end{aligned}$$

   for some fixed f. Similarly, $J_{\mathcal{X}}^{(n),\alpha}(x) \geq_1 J_{\omega^\omega \times \mathcal{X}}^{(n),\varnothing}(\alpha, x)$:

   $$\begin{aligned} e \in J_{\omega^\omega \times \mathcal{X}}^{(n),\varnothing}(\alpha, x) \Leftrightarrow H_{\Sigma_n^0}^{\omega^\omega \times \mathcal{X}}(\varnothing, e, \alpha, x) &\overset{H_{\Sigma_n^0,\varnothing,\alpha}^{\omega^\omega \times \mathcal{X}} \in \Sigma_n^{0,\alpha}(\omega \times \mathcal{X})}{\Leftrightarrow} H_{\Sigma_n^0}^{\omega \times \mathcal{X}}(\alpha, f, e, x) \\ &\Leftrightarrow H_{\Sigma_n^0}^{\mathcal{X}}(\alpha, S(f,e), x) \Leftrightarrow S(f,e) \in J_{\mathcal{X}}^{(n),\alpha}(x) \end{aligned}$$

2. For this statement we proceed by induction on ω proving that for $x \in \mathcal{X}$, $J_{\mathcal{X}}^{(n+1),\alpha}(x) \equiv_1 J \circ J_{\mathcal{X}}^{(n),\alpha}(x)$. Again $J_{\mathcal{X}}^{(n+1),\alpha}(x) \leq_1 J \circ J_{\mathcal{X}}^{(n),\alpha}(x)$ holds because:

$$\begin{aligned} e \in J_{\mathcal{X}}^{(n+1),\alpha}(x) &\Leftrightarrow H_{\Sigma^0_{n+1}}^{\mathcal{X}}(\alpha, e, x) \Leftrightarrow \exists i \in \omega(\neg H_{\Sigma^0_n}^{\omega\times\mathcal{X}}(\alpha, e, i, x)) \\ &\Leftrightarrow \exists i \in \omega(\neg H_{\Sigma^0_n}^{\mathcal{X}}(\alpha, S(e,i), x)) \\ &\Leftrightarrow \exists i \in \omega(S(e,i) \notin J_{\mathcal{X}}^{(n),\alpha}(x)) \end{aligned}$$

the last relation defines a $\Sigma_1^{0,J_{\mathcal{X}}^{(n),\alpha}(x)}(\omega)$ relation and hence it is one-one-reducible to its jump $J(J_{\mathcal{X}}^{(n),\alpha}(x))$. For the other direction, we observe that $e \in J \circ J_{\mathcal{X}}^{(n),\alpha}(x)$ defines a $\Sigma^{0,\alpha}_{n+1}$ relation on $\omega \times \mathcal{X}$ and hence for some fixed $f \in \omega$:

$$\begin{aligned} e \in J \circ J_{\mathcal{X}}^{(n),\alpha}(x) &\Leftrightarrow H_{\Sigma^0_{n+1}}^{\omega\times\mathcal{X}}(\alpha, f, e, x) \Leftrightarrow H_{\Sigma^0_{n+1}}^{\mathcal{X}}(\alpha, S(f,e), x) \\ &\Leftrightarrow S(f,e) \in J_{\mathcal{X}}^{(n+1),\alpha}(x) \end{aligned}$$

□

In light of this correspondence and the fact that the Σ^0_1-jump extends the usual one, we denote by x' the unrelativized Σ^0_1-jump (and $x^{(n)}$ the n-th unrelativized Σ^0_n-jump). In addition, this jump operator is well-defined on the structure of continuous degrees:

Lemma 2.40 ([GKN21, Lemma 2.9])**.** Given $\mathcal{X}$ and $\mathcal{Y}$ recursive spaces

$$\forall x \in X \forall y \in Y (x \leq_M y \Leftrightarrow x' \leq_1 y')$$

Proof. $\Rightarrow$ Let $f : \mathcal{Y} \rightharpoonup \mathcal{X}$ be the Σ^0_1-recursive function witnessing $x \leq_M y$. We prove that

Claim 2. There exists a computable injective function $S : \omega \to \omega$ such that

$$\forall y \in \mathrm{dom}(f) \forall e \in \omega(H_{\Sigma^0_1}^{\mathcal{X}}(\varnothing, e, f(y)) \Leftrightarrow H_{\Sigma^0_1}^{\mathcal{Y}}(\varnothing, S(e), y))$$

Proof of the Claim. Observe that f Σ^0_1-recursive on its domain means that for some semirecursive $D^* \subseteq \omega^2$:

$$\forall y \in \mathrm{dom}(f) \forall n \in \omega\big(f(y) \in V_n^{\mathcal{X}} \Leftrightarrow \exists m \in \omega(y \in V_m^{\mathcal{Y}} \wedge D^*(n,m))\big)$$

therefore for any $y \in \mathrm{dom}(f)$

$$\begin{aligned} H_{\Sigma^0_1}^{\mathcal{X}}(\varnothing, e, f(y)) &\Leftrightarrow \exists n \in W_e(f(y) \in V_n^{\mathcal{X}}) \\ &\Leftrightarrow \exists n, m \in \omega(n \in W_e \wedge D^*(n,m) \wedge y \in V_m^{\mathcal{Y}}) \end{aligned}$$

we now define:

$$h(e,m) = \begin{cases} 1 & \text{if } \exists n \in \omega(n \in W_e \wedge D^*(n,m)) \\ \uparrow & \text{otherwise} \end{cases}$$

such function is computable because its graph is Σ^0_1, hence $\varphi_j = h$ for some $j \in \omega$ and, by the S-m-n Theorem, there is an injective recursive function $\tilde{S} : \omega^2 \to \omega$ such that: $\varphi_j(e,m) = \varphi_{\tilde{S}(j,e)}(m)$. Hence, for any $y \in \mathrm{dom}(f)$ and any $e \in \omega$:

$$H^{\mathcal{Y}}_{\Sigma^0_1}(\varnothing, \tilde{S}(j,e), y) \Leftrightarrow \exists m \in W_{\tilde{S}(j,e)}(y \in V^{\mathcal{Y}}_m) \Leftrightarrow H^{\mathcal{X}}_{\Sigma^0_1}(\varnothing, e, f(y)) \quad \square$$

As $x = f(y)$ and by the claim $S : \omega \to \omega$ is injective, it follows that $x' \leq_1 y'$.[7]

$\Leftarrow$ Let $g : \omega \to \omega$ be the injective function witnessing that $x' \leq_1 y'$, recall that $\mathrm{N}_{\mathrm{base}}(x) = \{n \in \omega \mid x \in V^{\mathcal{X}}_n\}$. We observe that:

$$e \in \mathrm{N}_{\mathrm{base}}(x) \Leftrightarrow (x \in V^{\mathcal{X}}_e \wedge \{e\} = W_{h(e)}) \Leftrightarrow h(e) \in x'$$

where $h : \omega \to \omega$ is the computable function that associate to each finite set a code as semirecursive set (such function can be defined using the S-m-n Theorem see [Ter04, Exercise 2.5.17 and Proposition 3.4.2]). Therefore:

$$e \in \mathrm{N}_{\mathrm{base}}(x) \Leftrightarrow g \circ h(e) \in y' \Leftrightarrow W_{g \circ h(e)} \cap \mathrm{N}_{\mathrm{base}}(y) \neq \varnothing$$

In particular this implies that $\mathrm{N}_{\mathrm{base}}(x) \leq_e \mathrm{N}_{\mathrm{base}}(y)$, and hence $x \leq_M y$ by Fact 1. $\square$

Lemma 2.41 ([GKN21, Lemma 2.10])**.** Given $\mathcal{X}$ recursive space

$$\forall x \in X \exists p \in \omega^\omega (\rho_{\mathcal{X}}(p) = x \wedge x' \equiv_T p')$$

Proof. We proceed by the finite extension method[8] constructing $p = \bigcup_{s \in \omega} p_s \in \omega^\omega$ such that $\mathrm{ran}(p) = \mathrm{N}_{\mathrm{base}}(x)$ by a x'-computable way.
Let $p_0 = \varepsilon$ be the empty string. At stage $s \in \omega$, suppose we have $p_s \in \omega^{<\omega}$, then (using x' as oracle) we check if x is in the Σ^0_1 set defined as $\bigcup_{\tau \in \omega^{<\omega}} \{\bigcap_{i \in \mathrm{ran}(\tau)} V^{\mathcal{X}}_i \mid p_s \prec \tau \wedge \varphi^\tau_s(s) \downarrow\}$. We consider two steps for defining the new element:

- if x is in such a set and $\tau \succ p_s$ is a witness of $x \in \bigcap_{i \in \mathrm{ran}(\tau)} V^{\mathcal{X}}_i$ then we set $p^*_s \coloneqq \tau$, otherwise $p^*_s \coloneqq p_s$

[7] Considering the notation that we introduce in Chapter 3, using the same technique one can prove that $f^{-1}\,\mathbf{\Sigma}^0_1 \subseteq \mathbf{\Sigma}^0_1$ holds recursive-uniformly (see Proposition 3.9).

[8] For an explanation and some classical applications of the finite extension method we refer the reader to [MP22, Chapter 4 Section 8].

- if $x \in V_s^{\mathcal{X}}$ then $p_{s+1} \coloneqq p_s^{*\frown} s$, otherwise $p_{s+1} \coloneqq p_s^*$.

Now we prove that $\operatorname{ran}(p) = \mathrm{N}_{\text{base}}(x)$:

$\subseteq$ We observe that in the first step of the construction of p_{s+1}, $\operatorname{ran}(p_s) \subseteq \mathrm{N}_{\text{base}}(x)$ implies that $\operatorname{ran}(\tau) \subseteq \mathrm{N}_{\text{base}}(x)$ (because $x \in \bigcap_{i \in \operatorname{ran}(\tau)} V_i^{\mathcal{X}}$) and in the second one we extend it maintaining this condition. Thus, this inclusion follows by induction.

$\supseteq$ If $s \in \mathrm{N}_{\text{base}}(x)$ then $s \in \operatorname{ran}(p_{s+1})$ for the second step.

In conclusion, we have:

$$p'(s) = 1 \Leftrightarrow \varphi_s^p(s) \downarrow \Leftrightarrow \varphi_s^{p_{s+1}}(s) \downarrow$$

Indeed, if $\varphi_s^{p_{s+1}}(s) \uparrow$ this means that no witness τ is found at stage s and hence, as $\operatorname{ran}(p) = \mathrm{N}_{\text{base}}(x)$, we have $\varphi_s^p(s) \uparrow$. □

2.2.2 The Shore Slaman Join Theorem

We now prove the main result of this section: the Shore Slaman Join Theorem for continuous degrees. This result is the backbone on which are built the applications to Descriptive Set Theory that we present in Chapter 3. The proof we give is an "expanded" version of the one in [GKN21]. With "expanded" we mean that we make explicit all the details omitted in the article merging their proof with the original one for Turing degrees in [SS99]. Following the approach in [GKN21], before starting the proof we need to restate some classical results about Π_1^0-classes (that is sets in $\Pi_1^0(2^\omega)$) to the context of recursive spaces.

Definition 2.42. Given an oracle $\varepsilon \in \omega^\omega$ and a recursive space $\mathcal{Z}$, a point $z \in \mathcal{Z}$ is ε**-low** if $(z \oplus \varepsilon)' \leq_T \varepsilon'$.

Proposition 2.43 (Low Basis Theorem [GKN21])**.** Given an effectively compact space $\mathcal{H}$ and an oracle $\varepsilon \in \omega^\omega$. If $P \in \Pi_1^{0,\varepsilon}(\mathcal{H})$ is nonempty, then exists an ε-low point $z \in P$.

Proof. We construct a decreasing sequence $(Q_e)_{e \in \omega}$ of sets in $\Pi_1^{0,\varepsilon}(\mathcal{H})$ in a way that the corresponding sequence of indices is ε'-computable.
We define $Q_0 \coloneqq P$. Then, having Q_e we define:

$$Q_{e+1} \coloneqq \begin{cases} Q_e & \text{if } Q_e \subseteq G^{\mathcal{H}}_{\Sigma_1^0, e^\frown \varepsilon} \\ Q_e \setminus G^{\mathcal{H}}_{\Sigma_1^0, e^\frown \varepsilon} & \text{otherwise} \end{cases}$$

Since Q_e is effectively closed, by Proposition 2.24, having the indices of Q_e, we can decide whether $Q_e \subseteq G^{\mathcal{H}}_{\Sigma^0_1, e^\frown \varepsilon}$ in an ε'-computable manner (uniformly in e). Finally, we notice that given $z \in Q := \bigcap_{e \in \omega} Q_e$:

$$J^{\varepsilon}_{\mathcal{H}}(z)(e) = 1 \Leftrightarrow H^{\mathcal{H}}_{\Sigma^0_1}(\varepsilon, e, z) \Leftrightarrow z \in G^{\mathcal{H}}_{\Sigma^0_1, e^\frown \varepsilon} \overset{z \in Q}{\Leftrightarrow} Q_e \subseteq G^{\mathcal{H}}_{\Sigma^0_1, e^\frown \varepsilon}$$

and since the latter term is ε'-computable, thanks to Lemma 2.39 we get $(z \oplus \varepsilon)' \equiv_M J^{\varepsilon}_{\mathcal{H}}(z) \leq_T \varepsilon'$ □

Before proving the next result, we observe that, given a recursive space $\mathcal{X}$ and a recursive Polish space $\mathcal{Y}$, the e-th partial Σ^0_1-recursive function $\Phi^{\mathcal{X},\mathcal{Y}}_e$, being the biggest function induced by the e-th semirecursive set on ω, is defined on a Π^0_2 domain (uniformly in e), that is $\text{dom}(\Phi^{\mathcal{X},\mathcal{Y}}_e) \in \Pi^0_2(\mathcal{X})$. In fact, a more general result holds:

Theorem 2.44 ([Lou19, Theorem 3.4.7])**.** Given $\tilde{X}$ subspace of a recursive space $\mathcal{X}$, $\mathcal{Y}$ recursive Polish space, and $f : \tilde{X} \to \mathcal{Y}$ Σ^0_n-recursive. Then f can be extended to a Σ^0_n-recursive function $\hat{f} : \hat{X} \to \mathcal{Y}$ with $\tilde{X} \subseteq \hat{X} \in \Pi^0_{n+1}(\mathcal{X})$.

We don't prove the previous theorem but we point that the assumption of a Polish recursive codomain (and hence recursively isomorphic to a recursively presented Polish space) is essential. We recall that:

$$A \in \Pi^0_2(\mathcal{X}) \Leftrightarrow A = \bigcap_{m \in \omega} \bigcup_{n \in \omega} \{V^{\mathcal{X}}_n \mid A^*(m,n)\} \quad \text{for some } A^* \in \Sigma^0_1(\omega^2)$$

Therefore, we have $\text{dom}(\Phi^{\mathcal{X},\mathcal{Y}}_e) = \bigcap_{t \in \omega} D^e_t$ where $D^e_t = \bigcup_{n \in \omega} \{V^{\mathcal{X}}_n \mid D^*(t,n,e)\} \in \Sigma^0_1(\mathcal{X})$ (i.e. D^* is semirecursive), moreover we observe that $\{(e,t,x) \mid x \in D^e_t\} \in \Sigma^0_1(\omega \times \omega \times \mathcal{X})$.

Lemma 2.45 (Cone avoidance [GKN21, Lemma 2.11])**.** Given a recursive Polish space $\mathcal{Y}$ and an effectively compact space $\mathcal{H}$, then

$$\forall \varepsilon \in \omega^\omega \forall y \in \mathcal{Y}(y \nleq_M \varepsilon \Rightarrow \forall P \in \Pi^{0,\varepsilon}_1(\mathcal{H})(P \neq \varnothing \Rightarrow \exists z \in P(y \nleq_M z \wedge z \leq_M y \oplus \varepsilon')))$$

Proof. Without loss of generality, we may assume that $\mathcal{Y}$ is a recursively presented Polish space $(Y, \mathrm{d}_{\mathcal{Y}}, \boldsymbol{r}_{\mathcal{Y}})$, this is because $\mathcal{Y}$ is recursively isomorphic to such a space. By almost totality Lemma 2.35, either $\varepsilon' \leq_M y$ or $y \oplus \varepsilon'$ is total. Thus, there are two cases:

1. If $\varepsilon' \leq_M y$, then by the previous proposition there exists an ε-low point $z \in P$ and such point satisfies the thesis. Indeed:
   - $z \leq_M (z \oplus \varepsilon)' \leq_T \varepsilon' \leq_M y \oplus \varepsilon'$.

- Assume towards a contradiction that $y \leq_M z$, then $z' \leq_T (z \oplus \varepsilon)' \leq \varepsilon' \leq_M y \leq_M z$ that is impossible. $\lightning$

2. If $y \oplus \varepsilon'$ is total, by Point 5 of Fact 1, there is a canonical name $p_y \in \omega^\omega$ of y (that is, such that $p_y \oplus \varepsilon' \equiv_M y \oplus \varepsilon'$). We define a sequence of $\Pi^{0,\varepsilon}_1$ sets contained in P starting from $P_0 := P$. Having $P_e \subseteq P$, we describe how to define P_{e+1}. We consider the sets D^e_t given by the application of Theorem 2.44.

   Claim 3. If $P_e \subseteq \bigcap_{t \in \omega} D^e_t$, then $\exists z \in P_e(\Phi^{\mathcal{H},\mathcal{Y}}_e(z) \neq y)$.

   Proof of the Claim. Since $P_e \subseteq \bigcap_{t \in \omega} D^e_t = \mathrm{dom}(\Phi^{\mathcal{H},\mathcal{Y}}_e)$ then we have $\forall z \in P_e(\Phi^{\mathcal{H},\mathcal{Y}}_e(z) \in \mathcal{Y})$. Assume towards a contradiction that $\forall z \in P_e(\Phi^{\mathcal{H},\mathcal{Y}}_e(z) = y)$. By ε-effective compactness, for any $i \in \omega$, one can ε-computably find a finite set $V_i \subseteq W_e \cap \{\langle m, n\rangle \mid q_{(m)_1} < 2^{-i}\}$ such that:
   $$P_e \subseteq \bigcup_{\langle m,n\rangle \in V_i} V^{\mathcal{H}}_n \quad \wedge \quad \bigcap_{\langle m,n\rangle \in V_i} V^{\mathcal{Y}}_m \neq \varnothing$$
   (indeed the first condition is $\Sigma^{0,\varepsilon}_1(\omega^3)$ while the second is $\Sigma^0_1(\omega^2)$). Therefore, the diameter of $\bigcap_{\langle m,n\rangle \in V_i} V^{\mathcal{Y}}_m$ is smaller than 2^{-i+1}, and it gives an ε-computable decreasing sequence of open sets converging to y. However, this contradicts our assumption that $y \nleq_M \varepsilon$. $\lightning$ $\square$

   Moreover, observe that the ε-effective compactness of $\mathcal{H}$ (by Proposition 2.24) allows us to decide in an ε'-computable manner whether $P_e \subseteq D^e_t$ or not. Therefore, by the previous claim, either $P_e \nsubseteq D^e_t$ for some $t \in \omega$ or $\exists z \in P_e(\Phi^{\mathcal{H},\mathcal{Y}}_e(z) \neq y)$. In particular, in the latter case, there is $\langle m, n\rangle \in W_e$ witnessing that $\Phi^{\mathcal{H},\mathcal{Y}}_e[V^{\mathcal{H}}_n] \subseteq V^{\mathcal{Y}}_m$, $z \in V^{\mathcal{H}}_n$ and $y \notin \hat{V}^{\mathcal{Y}}_m$ (where $\hat{V}^{\mathcal{Y}}_m$ is the closure of the ball $V^{\mathcal{Y}}_m$). Therefore, for a sufficiently large $t \in \omega$, we have
   $$\hat{V}^{\mathcal{H}}_{\langle (n)_0, r(t)\rangle} \cap P_e \neq \varnothing \wedge V^{\mathcal{Y}}_{p_y(t)} \cap V^{\mathcal{Y}}_m = \varnothing$$
   where $r(t)$ is the index in our fixed enumeration of $\mathbb{Q}^+$ that correspond to $q_{(n)_1} - 2^{-t}$ (i.e. $q_{r(t)} = q_{(n)_1} - 2^{-t}$) and $\hat{V}^{\mathcal{H}}_n$ is the closure of the ball $V^{\mathcal{H}}_n = B^{\mathcal{H}}(\boldsymbol{r}((n)_0), q_{(n)_1})$ in the effective basis of $\mathcal{H}$. Again, by ε-effective compactness we can decide ε'-computably if $\hat{V}^{\mathcal{H}}_l \cap P_e \neq \varnothing$ (because $\hat{V}^{\mathcal{H}}_l \cap P_e = \varnothing \Leftrightarrow \hat{V}^{\mathcal{H}}_l \subseteq H \smallsetminus P_e$) for the closure $\hat{V}^{\mathcal{H}}_l$ in H of the ball $V^{\mathcal{H}}_l$. Hence we can use $p_y \oplus \varepsilon'$ to decide if such condition holds. Thus:

   - If $P_e \nsubseteq D^e_t$ for some t, we set $Q_{e+1} := P_e \smallsetminus D^e_t$, in this way we ensure that $Q_{e+1} \cap \mathrm{dom}(\Phi^{\mathcal{H},\mathcal{Y}}_e) = \varnothing$.

- Otherwise, we define $Q_{e+1} := P_e \cap \hat{V}^{\mathcal{H}}_{\langle (n)_1, r(t) \rangle}$, and ensure that $y \notin \Phi_e[Q_{e+1}] \subseteq V^{\mathcal{Y}}_m$.

Then, we find the first closed ball $\hat{B}$ (w.r.t the enumeration of the basis) of radius 2^{-e-2} such that $\hat{B} \cap Q_{e+1} \neq \emptyset$ (this can be done ε'-computably again by the ε-effective compactness) and set $P_{e+1} := \hat{B} \cap Q_{e+1}$.

By compactness, as $\operatorname{diam}(P_e) \underset{e \to \infty}{\longrightarrow} 0$, there is a $z \in H$ such that $\{z\} = \bigcap_{e \in \omega} P_e$.We observe that the whole construction is computable in $p_y \oplus \varepsilon' \equiv_M y \oplus \varepsilon'$. In particular, the oracle $p_y \oplus \varepsilon'$ is not only able to decide a sequence of indices for $\{P_e \mid e \in \omega\}$, but also a sequence of indices for the set $\{B_j \mid j \in \omega\}$ of open balls of the basis such that the radius of B_j is 2^j and $P_j \subseteq B_j$, which yields a name for z. Therefore: $z \leq_M p_y \oplus \varepsilon' \equiv_M y \oplus \varepsilon'$. Moreover, by our construction we have that for every $e \in \omega$ either $\Phi_e^{\mathcal{H},\mathcal{Y}}(z) \uparrow$ or $\Phi_e^{\mathcal{H},\mathcal{Y}}(z) \neq y$ and hence $y \not\leq_M z$. □

Theorem 2.46 (Shore-Slaman Join Theorem for recursive spaces [GKN21, Theorem 2.12])**.** Let $\mathcal{X}$ and $\mathcal{Y}$ be recursive spaces, $x \in X$, $y \in Y$, and $n \in \omega$. If $y \not\leq_M x^{(n)}$, then there is a $G \in 2^\omega$ such that $G \geq_M x \wedge G^{(n+1)} \equiv_M G \oplus y$. Furthermore:

- $y \oplus x^{(n+1)}$ is total $\Rightarrow G^{(n+1)} \equiv_M G \oplus y \equiv_M y \oplus x^{(n+1)}$
- $y \oplus x^{(n+1)}$ is not total $\Rightarrow G^{(n+1)} \equiv_M G \oplus y \equiv_M G \oplus x^{(n+1)}$.

Moreover, the same results holds for any ordinal $\xi < \omega_1^{\text{CK}}$, in particular: if $\forall \zeta < \xi (y \not\leq_M x^{(\zeta)})$, then $\exists G \in 2^\omega (G \geq_M x \wedge G^{(\xi)} \equiv_M G \oplus y)$.

The proof of Shore-Slaman Join Theorem is divided in two cases following the totality or not of the degree corresponding to $y \oplus x^{(n+1)}$ (for this reason we say that the proof is non-uniform). The total case (similarly to the original proof in [SS99]) uses the following transfinite version of the Friedberg Jump Inversion Theorem, proved by Jockusch and Shore in [JS84] for α-REA-operators (and so holds for the iterated Turing Jump $J^{(\alpha)}(x) = x^{(\alpha)}$).

Theorem 2.47 (Transfinite Relativized Friedberg Jump Inversion Theorem)**.** $\forall z \in \omega^\omega \forall \xi \leq \omega_1^z$ (the first non-computable ordinal relative to z) $\forall x \in \omega^\omega \exists y \in \omega^\omega (z \leq_T y \wedge y^{(\xi)} \equiv_T x \oplus z^{(\xi)})$

We now have all the prerequisites for proving Theorem 2.46. In particular, the proof is similar to the one in [SS99] except for the forcing conditions adapted to the non total case.

Proof. Without loss of generality we can assume that $x \in 2^\omega$ and $y \in [0,1]^\omega$ (more precisely $\mathcal{Y} = [0,1]^\omega$). The latter can be assumed because every recursive space is recursively embedded into the Hilbert cube $[0,1]^\omega$ (by Corollary 1.27), while the former assumption can be done because:

- If $n = 0$, since $y \nleq_M x$, by Corollary 2.34, there exists a $z \in 2^\omega$ such that $x \leq_M z$ and $y \nleq_M z$. By point 4. of Fact 1 there exists a name p_x of x, such that $x \leq_M p_x \leq_M z$. We thus consider a $z_x \in 2^\omega$ such that $z_x \equiv_T p_x$.[9] Therefore from the thesis for y and z_x we have a $G \geq_M z_x \geq_M x$ that satisfies the thesis also for x (because $x' \equiv_M p'_x$ by Lemma 2.41).

- If $n > 0$ we can choose any name $p \in \omega^\omega$ for x (and a Turing equivalent element in 2^ω) and conclude as in the previous case, since Lemma 2.41 implies that $\forall n \geq 1 (x^{(n)} \equiv_T p^{(n)})$.

We consider the recursively bounded tree $\boldsymbol{H} \subseteq \omega^{<\omega}$ of names for $[0,1]^\omega$ from Example 2.25 (and the respective representation $\kappa : [\boldsymbol{H}] \to [0,1]^\omega$).

Definition 2.48.

1. A **Turing functional on $\boldsymbol{H}$** is a set $\Phi \subseteq \omega \times 2 \times \boldsymbol{H}$ such that $\forall n \in \omega \forall k_1, k_2 \in 2 \forall \sigma_1, \sigma_2 \in \boldsymbol{H}((\sigma_1, \sigma_2 \text{ compatible } \wedge (n, k_1, \sigma_1) \in \Phi \wedge (n, k_2, \sigma_2) \in \Phi) \Rightarrow (k_1 = k_2 \wedge \sigma_1 = \sigma_2))$

2. A Turing functional on $\boldsymbol{H}$ Φ is **use monotone** if the following hold:

   (a) $\forall n_1, n_2 \in \omega \forall k_1, k_2 \in 2 \forall \sigma_1, \sigma_2 \in \boldsymbol{H}((\sigma_1 \precneqq \sigma_2 \wedge (n_1, k_1, \sigma_1) \in \Phi \wedge (n_2, k_2, \sigma_2) \in \Phi) \Rightarrow (n_1 < n_2))$

   (b) $\forall n_1, n_2 \in \omega \forall k_2 \in 2 \forall \sigma_2 \in \boldsymbol{H}((n_1 < n_2 \wedge (n_2, k_2, \sigma_2) \in \Phi) \Rightarrow \exists k_1 \in 2 \exists \sigma_1 \in \boldsymbol{H}(\sigma_1 \preceq \sigma_2 \wedge (n_1, k_1, \sigma_1) \in \Phi))$

Without loss of generality, we can consider a use monotone Turing functional Φ as an element of 2^ω.

Notation 2.49. Given a Turing functional on $\boldsymbol{H}$ Φ, we write:

1. $\Phi(\sigma)(n) = k$ if $\exists \tau \in \boldsymbol{H}$ such that $\tau \preceq \sigma \wedge (n, k, \tau) \in \Phi$

2. $\text{dom}(\Phi) = \{\sigma \in \boldsymbol{H} \mid \exists n \in \omega \exists k \in 2((n, k, \sigma) \in \Phi)\}$

We observe that an use monotone Turing functional Φ defines a partial monotone function from $\boldsymbol{H}$ to $2^{<\omega}$, therefore we can see it as a partial continuous function from $[\boldsymbol{H}]$ to 2^ω.

Definition 2.50. A Turing functional on $\boldsymbol{H}$ Φ is **consistent along a point** $z \in [0,1]^\omega$ if:

$$\forall \sigma, \tau \in \boldsymbol{H}\Big(z \in B^*_\sigma \cap B^*_\tau \Rightarrow \forall n \in \omega\big((\Phi(\sigma)(n) \downarrow \wedge \Phi(\tau)(n) \downarrow) \Rightarrow \Phi(\sigma)(n) = \Phi(\tau)(n)\big)\Big)$$

[9]This can be done since every degree corresponding to an element of ω^ω is total.

Even though a Turing functional in general is not Σ^0_1 set we have the following property for consistent Turing functionals along a point:

Claim 4. Given a use monotone Turing functional on $\boldsymbol{H}$ Φ, if it is consistent along a point $z \in [0,1]^\omega$ and there exists a κ-name of z said $\alpha_z \in \boldsymbol{H}$ such that $\Phi(\alpha_z) \in 2^\omega$ is defined, then $\Phi(\alpha_z) \leq_M \Phi \oplus z$.

Proof of the Claim. We describe a computable function from the names of $\Phi \oplus z$ to $\Phi(\alpha_z)$ (since 2^ω is 0-dimensional recursive and hence it recursively embeds into ω^ω, this is equivalent to define it for a name of $\Phi(\alpha_z)$).
In particular, given a name for Φ and a κ-name β_z for z, for every $n \in \omega$ we search (by brute-force) $(\sigma, m) \in \boldsymbol{H} \times 2$ such that $(n, m, \sigma) \in \Phi$. We observe that:

1. Since $\Phi(\alpha_z)(n)$ is defined there is at least one such pair $(\sigma, m) \in \boldsymbol{H} \times 2$, and hence the procedure halts.
2. We can recover Φ from any of its names because ρ_{2^ω} is computable.
3. Having the consistency over the point z, $\Phi(\alpha_z) = \Phi(\beta_z)$ and thus, to assure that $\Phi(\alpha_z)(n) = m$ we only have to check that $\Phi(\beta_z)(n) = m$. □

We briefly outline the proof that follows. It is a modification of the classical construction of Shore and Slaman in [SS99], but it is divided in two cases depending on $y \oplus x^{(n+1)}$ total or not, in particular:

- If $y \oplus x^{(n+1)}$ is total, then we can choose a canonical name $\alpha_y \in \boldsymbol{H}$ of y such that $\alpha_y \leq_T y \oplus x^{(n+1)}$ (therefore $\alpha_y \oplus x^{(n+1)} \equiv_T y \oplus x^{(n+1)}$). Thus, we can carry out a continuous analogue of the classical proof doing a construction computable in $\alpha_y \oplus x^{(n+1)}$ and obtaining $G^{(n+1)} \equiv_M G \oplus y \equiv_M y \oplus x^{(n+1)}$.
- If $y \oplus x^{(n+1)}$ is not total, this strategy does not work because different names for y produce different sets G. However, in this case the construction carried out is computable in $G \oplus x^{(n+1)}$ but for obtaining this the Kumabe-Slaman forcing conditions need to be modified including two additional parameters ($\lambda \in \omega^\omega$ and $\varepsilon \in \mathbb{Q}^+$). In particular, we construct a partial continuous function $\Phi_G : [\boldsymbol{H}] \rightharpoonup 2^\omega$ (using conditions similar to the usual Kumabe-Slaman forcing), then setting $G \coloneqq x \oplus \Phi_G \oplus \lambda_G$, and assuring that Φ_G is consistent along y, we have that $G \oplus y \geq_M \Phi_G(\alpha_y)$ for a priori fixed κ-name of y. To be precise, the modifications to the conditions to the Kumabe-Slaman forcing ensure that $\alpha_y \leq_T G \oplus x^{(n+1)}$ (and hence $G \oplus y \leq_M G \oplus x^{(n+1)}$) to do so we code the name α_y in λ_G in a way that $G \oplus x^{(n+1)}$ can computably recover it.

Definition 2.51. We say that $\sigma \in \omega^{<\omega}$ **meets** $\beta \in \omega^\omega$ if $B^*_{\beta \upharpoonright \ell(\sigma)} \cap B^*_\sigma \neq \varnothing$, and that σ **meets** $S \subseteq \omega^\omega$ if σ meets some $\beta \in S$.

Definition 2.52 (Modified Kumabe-Slaman Forcing)**.** Let $\mathbb{P}_{\mathrm{KS}}$ be the following partial order

1. The elements p of $\mathbb{P}_{\mathrm{KS}}$ are the quadruples $(\Phi_p, \mathbf{X}_p, \lambda_p, \varepsilon_p)$ in which:
   (a) $\Phi_p \subseteq \omega \times 2 \times \boldsymbol{H}$ is a finite use-monotone Turing functional on $\boldsymbol{H}$
   (b) $\mathbf{X}_p$ is a finite subset of κ-names in $[\boldsymbol{H}]$
   (c) $\lambda_p \in \omega^{<\omega}$ is a finite string
   (d) $\varepsilon_p \in \mathbb{Q}^+ \cap (0,1]$
2. Given two elements $p, q \in \mathbb{P}_{\mathrm{KS}}$, we say that q **is stronger than** p ($q \leq p$) if and only if:
   (a) $\Phi_p \subseteq \Phi_q$, $\mathbf{X}_p \subseteq \mathbf{X}_q$, $\lambda_p \leq \lambda_q$ and $\varepsilon_q \leq \varepsilon_p$
   (b) $\forall \sigma \in \mathrm{dom}(\Phi_q) \setminus \mathrm{dom}(\Phi_p) \forall \tau \in \mathrm{dom}(\Phi_p)(\ell(\sigma) > \ell(\tau))$
   (c) $\sum\{2^{-\ell(\sigma)} \mid \sigma \in \mathrm{dom}(\Phi_q) \setminus \mathrm{dom}(\Phi_p)\} + \varepsilon_q \leq \varepsilon_p$
   (d) any $\sigma \in \mathrm{dom}(\Phi_q) \setminus \mathrm{dom}(\Phi_p)$ does not meet $\mathbf{X}_p$

   In this case we call $\sum\{2^{-\ell(\sigma)} \mid \sigma \in \mathrm{dom}(\Phi_q) \setminus \mathrm{dom}(\Phi_p)\}$ the *amount added* by $\mathrm{dom}(\Phi_q)$.

Notation 2.53. Given a modified Kumabe-Slaman forcing condition $p = (\Phi_p, \mathbf{X}_p, \lambda_p, \varepsilon_p)$, we write p^0 for $(\Phi_p, \varnothing, \lambda_p, \varepsilon_p)$.

As $\varnothing \subseteq \mathbf{X}_p$, then $p \leq p^0$. We observe that if $F \subseteq P$ is a sufficiently $\mathbb{P}_{\mathrm{KS}}$-generic filter, then F is associated to the functional $\Phi_F = \bigcup\{\Phi_p \mid p \in F\}$ and the string $\lambda_F = \bigcup_{p \in F} \lambda_p$. To this extent we construct a decreasing sequence of conditions $(p_n)_{n \in \omega}$, and then define $G := x \oplus \Phi_G \oplus \lambda_G \in 2^\omega$ where $\Phi_G := \bigcup_{n \in \omega} \Phi_p$ and $\lambda_G := \bigcup_{n \in \omega} \lambda_{p_n}$.[10] We use the symbol $\dot{r}_{gen}$ to denote the generic element $\Phi_G \oplus \lambda_G$. In the following, we treat $\Phi_G \oplus \lambda_G$ as subset of ω (i.e. as element in 2^ω) without explicitly expressing the coding apparatus needed to represent it in this way.
We recall the definition of the forcing relation and frame in it for $\mathbb{P}_{\mathrm{KS}}$.

Definition 2.54 (The forcing relation $\Vdash_{\mathbb{P}_{\mathrm{KS}}}$ [DM22, Definition 7.4.1])**.** Let $\mathcal{L}$ be the language of second order arithmetic i.e. including equality =, inequality <, sum +, and product $\cdot$ for numbers, and with a membership symbol $\in$ (to be used only in formulas of the kind "$n \in A$" where n number term and A set term), augmented with a number constant n for each number $n \in \omega$ and a set constant x (corresponding to the fixed element $x \in 2^\omega$). Let $\mathcal{L}(\dot{r}_{gen})$

[10] To be precise, the defined G is in ω^ω, but we can always consider a Turing equivalent element in 2^ω.

be $\mathcal{L}$ augmented with a set constant $\dot{r}_{gen}$ (for the generic element $\Phi_G \oplus \lambda_G$). Given a sentence $\varphi \in \mathcal{L}(\dot{r}_{gen})$, $p \in \mathbb{P}_{\mathrm{KS}}$ **forces** φ ($p \Vdash_{\mathbb{P}_{\mathrm{KS}}} \varphi$) is inductively defined as follows:

1. $p \Vdash_{\mathbb{P}_{\mathrm{KS}}} \varphi \Leftrightarrow \varphi$ is true, when φ is atomic in $\mathcal{L}$
2. $p \Vdash_{\mathbb{P}_{\mathrm{KS}}} n \in \dot{r}_{gen} \Leftrightarrow \Phi_p \oplus \lambda_p(n) = 1$ and $n < \ell(\Phi_p \oplus \lambda_p)$
3. $p \Vdash_{\mathbb{P}_{\mathrm{KS}}} \exists x < m\varphi(x) \Leftrightarrow$ for some $n \in \omega(n < m \wedge p \Vdash_{\mathbb{P}_{\mathrm{KS}}} \varphi(n))$
4. $p \Vdash_{\mathbb{P}_{\mathrm{KS}}} \neg\varphi \Leftrightarrow \forall q \le p \neg(q \Vdash_{\mathbb{P}_{\mathrm{KS}}} \varphi)$
5. $p \Vdash_{\mathbb{P}_{\mathrm{KS}}} (\varphi \vee \psi) \Leftrightarrow p \Vdash_{\mathbb{P}_{\mathrm{KS}}} \varphi$ or $p \Vdash_{\mathbb{P}_{\mathrm{KS}}} \psi$
6. $p \Vdash_{\mathbb{P}_{\mathrm{KS}}} \exists x\varphi(x) \Leftrightarrow$ for some $n \in \omega$ $p \Vdash_{\mathbb{P}_{\mathrm{KS}}} \varphi(n)$

Definition 2.55. Given $p \in \mathbb{P}_{\mathrm{KS}}$, and $\psi(x \oplus \dot{r}_{gen})$ be a sentence of the form $\forall m\theta(m, x \oplus \dot{r}_{gen})$, a sequence of strings $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_k)$ in $\boldsymbol{H}^k$ all of the same length (i.e. $\ell(\tau_i) = \ell(\tau_j)$ for every $1 \le i, j \le k$), we say that $\boldsymbol{\tau}$ is **essential to (force the sentence)** $\neg\psi(x \oplus \dot{r}_{gen})$ **over** $\Phi_p \oplus \lambda_p$ if: for any condition $q \in \mathbb{P}_{\mathrm{KS}}$

$$q \le p^0 \wedge \exists m \in \omega(q \Vdash_{\mathbb{P}_{\mathrm{KS}}} \neg\theta(m, x \oplus \dot{r}_{gen})) \Rightarrow \\ \exists \sigma \in \mathrm{dom}(\Phi_q) \smallsetminus \mathrm{dom}(\Phi_p) \exists j \le k(\sigma \text{ meets } \tau_j)$$

Recall that, in the Hilbert cube it can be computably decided whether or not two basic open balls (with respect to $\boldsymbol{H}$) intersect.

Definition 2.56. Given a condition $p \in \mathbb{P}_{\mathrm{KS}}$ and $k \in \omega$, we define

$$T(p, \psi, k) \coloneqq \{\boldsymbol{\tau} \in \boldsymbol{H}^k \mid \boldsymbol{\tau} \text{ is essential to } \neg\psi(x \oplus \dot{r}_{gen}) \text{ over } \Phi_p \oplus \lambda_p\}$$

and order it by extension on all coordinates (i.e. $\boldsymbol{\tau} \trianglelefteq \boldsymbol{\sigma}$ iff $\forall i \le k(\tau_i \le \sigma_i)$).

We observe that $(T(p, \psi, k), \trianglelefteq)$ is a subtree of the tree of elements in $\boldsymbol{H}^k$ with coordinates of the same lenght ordered by $\trianglelefteq$. Indeed, if $\boldsymbol{\tau} \trianglelefteq \boldsymbol{\sigma}$ and $\boldsymbol{\sigma}$ is essential to $\neg\psi(x \oplus \dot{r}_{gen})$ over $\Phi_p \oplus \lambda_p$ then so is $\boldsymbol{\tau}$ because the diameter of the last digit of any of its component is grater (or equal) than the ones of $\boldsymbol{\sigma}$. Moreover, $(T(p, \psi, k), \trianglelefteq)$ is a recursively bounded recursive tree (since $\boldsymbol{H}$ is so).

Lemma 2.57 ([SS99, Lemma 2.10])**.** Given a condition $p \in \mathbb{P}_{\mathrm{KS}}$, and a Π^0_n sentence $\psi(x \oplus \dot{r}_{gen})$ with $n \ge 1$ and $k \in \omega$, then:

1. Given $\beta_1, \ldots, \beta_k \in [\boldsymbol{H}]$ and $\mathbf{X} = \{\beta_1, \ldots, \beta_k\}$, if

$$(\Phi_p, \mathbf{X}, \lambda_p, \varepsilon_p) \Vdash_{\mathbb{P}_{\mathrm{KS}}} \psi(x \oplus \dot{r}_{gen})$$

   then $T(p, \psi, k)$ is infinite.

2. $T(p,\psi,k)$ is infinite, then it has an infinite path.
   Moreover, each infinite path $\overline{Y}$ in $T(p,\psi,k)$ is identified with a set of size k $\mathbf{X}(\overline{Y}) \subseteq [\boldsymbol{H}]$ such that $(\Phi_p, \mathbf{X}(\overline{Y}), \lambda_p, \varepsilon_p) \Vdash_{\mathbb{P}_{\mathrm{KS}}} \psi(x \oplus \dot{r}_{gen})$

Proof. Let $\psi(x \oplus \dot{r}_{gen})$ be of the form $\forall m \theta(m, x \oplus \dot{r}_{gen})$ with $\theta(m, x \oplus \dot{r}_{gen})$ Σ^0_{n-1} formula.

1. Given a set $\mathbf{X} = \{\beta_1, \dots, \beta_k\}$ such that $(\Phi_p, \mathbf{X}, \lambda_p, \varepsilon_p) \Vdash_{\mathbb{P}_{\mathrm{KS}}} \psi(x \oplus \dot{r}_{gen})$. We consider the set of the sequences $\boldsymbol{\tau}^l = (\beta_1 \upharpoonright l, \dots, \beta_k \upharpoonright l)$ for every $l \in \omega$. For all $q \in \mathbb{P}_{\mathrm{KS}}$ such that $q \leq p^0 \wedge \exists m \in \omega (q \Vdash_{\mathbb{P}_{\mathrm{KS}}} \neg\theta(m, x \oplus \dot{r}_{gen}))$ we have that q and $(\Phi_p, \mathbf{X}, \lambda_p, \varepsilon_p)$ are incompatible because any other condition stronger than them forces $\psi(x \oplus \dot{r}_{gen})$ and $\neg\psi(x \oplus \dot{r}_{gen})$, and so $(\Phi_q, \mathbf{X}_q \cup \mathbf{X}, \lambda_q, \varepsilon_q) \nleq (\Phi_p, \mathbf{X}, \lambda_p, \varepsilon_p)$. Therefore, there is an $i \leq k$ and exists $\sigma \in \mathrm{dom}(\Phi_q) \setminus \mathrm{dom}(\Phi_p)$ such that β_i meets σ. In particular, σ meets $\boldsymbol{\tau}^l_i$ for each $l \in \omega$, and this means that it is essential to $\neg\psi(x \oplus \dot{r}_{gen})$ over $\Phi_p \oplus \lambda_p$ and hence $T(p,\psi,k)$ is infinite.

2. By König's Lemma, since $T(p,\psi,k)$ is finitely branching, there is an infinite branch $\overline{Y}$. Let $\mathbf{X}(\overline{Y}) = \{\beta_1 \dots, \beta_k\}$ be the set in which each $\beta_i \in [\boldsymbol{H}]$ is the union of the i-th coordinate of the elements of $\overline{Y}$. For every $q \leq p^0$ and such that $\exists m \in \omega (q \Vdash_{\mathbb{P}_{\mathrm{KS}}} \neg\theta(m, x \oplus \dot{r}_{gen}))$ there is a $\sigma \in \mathrm{dom}(\Phi_q) \setminus \mathrm{dom}(\Phi_p)$ that meets at least one component of each element of $\overline{Y}$ (and hence of $\mathbf{X}(\overline{Y})$).

   Claim 5. $\forall m \in \omega \forall q \leq (\Phi_p, \mathbf{X}(\overline{Y}), \lambda_p, \varepsilon_p)(q \nVdash_{\mathbb{P}_{\mathrm{KS}}} \neg\theta(m, x \oplus \dot{r}_{gen}))$

   Proof of the Claim. Suppose towards a contradiction that there are an $m \in \omega$ and a $q \leq (\Phi_p, \mathbf{X}(\overline{Y}), \lambda_p, \varepsilon_p)$ such that $q \Vdash_{\mathbb{P}_{\mathrm{KS}}} \neg\theta(m, x \oplus \dot{r}_{gen})$. Then we have that $q \leq p^0$ (because $(\Phi_p, \mathbf{X}(\overline{Y}), \lambda_p, \varepsilon_p) \leq p^0$) and so $\exists \sigma \in \mathrm{dom}(\Phi_q) \setminus \mathrm{dom}(\Phi_p)$ such that σ meets $\mathbf{X}(\overline{Y})$ and this is impossible because $q \leq (\Phi_p, \mathbf{X}(\overline{Y}), \lambda_p, \varepsilon_p)$. ↯ □

   Therefore, $(\Phi_p, \mathbf{X}(\overline{Y}), \lambda_p, \varepsilon_p) \Vdash_{\mathbb{P}_{\mathrm{KS}}} \psi(x \oplus \dot{r}_{gen})$ (by the definition of the forcing relation for negated sentences). □

Lemma 2.58 ([SS99, Lemma 2.11])**.** Given a condition $p \in \mathbb{P}_{\mathrm{KS}}$, a Π^0_{n+1} sentence $\psi(x \oplus \dot{r}_{gen})$, and $k \in \omega$, then $T(p,\psi,k)$ is a $\Pi^{0,x}_{n+1}$ (hence $\Pi^{0,x^{(n)}}_1$ [11]) subtree of $(\boldsymbol{H}^k)^{<\omega}$ uniformly in Φ_p, λ_p, ε_p, ψ, and k.

Proof. Before starting the proof we observe that if $\neg\theta(x \oplus \dot{r}_{gen})$ is a bounded sentence about $x \oplus \dot{r}_{gen}$ (that is a sentence obtained by connectives and bounded quantifications), whether $(\Phi_p, \mathbf{X}_p, \lambda_p, \varepsilon_p) \Vdash_{\mathbb{P}_{\mathrm{KS}}} \neg\theta(x \oplus \dot{r}_{gen})$ holds or not is a bounded property, decidable uniformly in terms of Φ_p, $\mathbf{X}_p$, λ_p, ε_p,

[11] This follows from [MP22, Theorem 5.5]

x, and $\neg\theta(x \oplus \dot{r}_{gen})$ (this follows from the fact that the forcing relation $\Vdash_{\mathbb{P}_{\text{KS}}}$ is defined inductively and in the second clause, for sentences of the form $n \in \dot{r}_{gen}$, the condition is bounded w.r.t. p). Moreover, said $b \in \omega$ the bound on the quantifiers in the formula which defines such property, we observe that: given $\mathbf{X} \subseteq \mathbf{X}_p$ such that $\forall \alpha \in \mathbf{X}_p \exists \beta \in \mathbf{X}(\alpha \upharpoonright b = \beta \upharpoonright b)$ then

$$(\Phi_p, \mathbf{X}_p, \lambda_p, \varepsilon_p) \Vdash_{\mathbb{P}_{\text{KS}}} \neg\theta(x \oplus \dot{r}_{gen}) \Leftrightarrow (\Phi_p, \mathbf{X}, \lambda_p, \varepsilon_p) \Vdash_{\mathbb{P}_{\text{KS}}} \neg\theta(x \oplus \dot{r}_{gen})$$

$\Leftarrow$ it is true because $p = (\Phi_p, \mathbf{X}_p, \lambda_p, \varepsilon_p)$ is stronger.

$\Rightarrow$ Suppose towards a contradiction that

$$(\Phi_p, \mathbf{X}, \lambda_p, \varepsilon_p) \nVdash_{\mathbb{P}_{\text{KS}}} \neg\theta(x \oplus \dot{r}_{gen})$$

therefore $\exists l \leq (\Phi_p, \mathbf{X}, \lambda_p, \varepsilon_p)$ such that $l \Vdash_{\mathbb{P}_{\text{KS}}} \theta(x \oplus \dot{r}_{gen})$. Therefore, for any $\sigma \in \text{dom}(\Phi_l) \setminus \text{dom}(\Phi_p)$ σ does not meet $\mathbf{X}$. In particular, for the property of $\mathbf{X}$, we have also that for any $\sigma \in \text{dom}(\Phi_l) \setminus \text{dom}(\Phi_p)$ σ does not meet $\mathbf{X}$. Thus $l \leq p$, and hence it forces both $\neg\theta(x \oplus \dot{r}_{gen})$ and $\theta(x \oplus \dot{r}_{gen})$ a contradiction. $\lightning$.

Since there are only finitely many incompatible sequences in $\boldsymbol{H}$ of length b, we can "capture the possible behaviors" of the sets $\mathbf{X}_p$ by quantifying over the possible behaviors of subsets of the set $\{\eta \in \boldsymbol{H} \mid \ell(\eta) = b\}$. Therefore, deciding if there exists a finite set $\mathbf{X} \subseteq [\boldsymbol{H}]$ such that $(\Phi_p, \mathbf{X}, \lambda_p, \varepsilon_p) \Vdash_{\mathbb{P}_{\text{KS}}} \neg\theta(x \oplus \dot{r}_{gen})$ is a bounded property decidible uniformly in Φ_p, λ_p, ε_p, x, and $\neg\theta(x \oplus \dot{r}_{gen})$. Now we are ready to prove our lemma by induction on n:

Base case: Suppose that $n = 1$, so ψ is of the form $\forall z \theta(z, x \oplus \dot{r}_{gen})$ where $\theta(z, x \oplus \dot{r}_{gen})$ is a bounded formula. Let k be fixed and suppose that $\boldsymbol{\tau} \in \boldsymbol{H}^k$ with all the components with same length. By definition $\boldsymbol{\tau} \in T(p, \psi, k)$ if and only if for any condition $q \in \mathbb{P}_{\text{KS}}$:

$$q \leq p^0 \wedge \exists m \in \omega(q \Vdash_{\mathbb{P}_{\text{KS}}} \neg\theta(m, x \oplus \dot{r}_{gen})) \Rightarrow \\ \exists \sigma \in \text{dom}(\Phi_q) \setminus \text{dom}(\Phi_p) \exists j \leq k(\sigma \text{ meets } \tau_j)$$

For each suitable Φ_q, λ_q and ε_q, by the analysis of the forcing relation for bounded sentences done above, deciding whether there is a finite set $\mathbf{X}_q$ such that $q \leq p^0$ and $q \Vdash_{\mathbb{P}_{\text{KS}}} \neg\theta(m, x \oplus \dot{r}_{gen})$ is a bounded property of $\Phi_q \oplus \lambda_q$ and m. Therefore, the quantifier over $q \in \mathbb{P}_{\text{KS}}$ can be replaced by a quantifier over Φ_q, λ_q and ε_q with $q^0 \leq p^0$. Consequently, the fact that $\boldsymbol{\tau}$ is essential to (force the sentence) $\neg\psi(x \oplus \dot{r}_{gen})$ over $\Phi_p \oplus \lambda_p$ is a $\Pi^{0,x}_1$ property of $\boldsymbol{\tau}$, and so $T(p, \psi, k)$ is a $\Pi^{0,x}_1$ subtree and such definition is uniformly in terms of Φ_p, λ_p, ε_p, ψ, and k.

Inductive step: Suppose that the thesis holds for n and let $\psi(x \oplus \dot{r}_{gen})$ be a Π^0_{n+1} sentence of the form $\forall m \theta(m, x \oplus \dot{r}_{gen})$ with $\theta(m, x \oplus \dot{r}_{gen})$

Σ^0_n. As above, let k be fixed and suppose that $\boldsymbol{\tau} \in \boldsymbol{H}^k$ with all the components of same length. Again $\boldsymbol{\tau} \in T(p, \psi, k)$ if and only if for all Φ_q, λ_q, ε_q with $q^0 \leq p^0$ for all $m \in \omega$ if there is a set $\mathbf{X}_q$ of size k such that and $q \Vdash_{\mathbb{P}_{\text{KS}}} \neg\theta(m, x \oplus \dot{r}_{gen})$, then $\exists \sigma \in \text{dom}(\Phi_q) \setminus \text{dom}(\Phi_p) \exists j \leq k(\sigma$ meets $\tau_j)$. By the previous lemma, we have that:

"there is a set $\mathbf{X}_q$ of size k such that and $q \Vdash_{\mathbb{P}_{\text{KS}}} \neg\theta(m, x \oplus \dot{r}_{gen})$"

is equivalent to "$T(q, \neg\theta(m, x \oplus \dot{r}_{gen}), k)$ is infinite". Since $\neg\theta(m, x \oplus \dot{r}_{gen})$ is a $\Pi^{0,x}_n$ sentence, we can apply the inductive hypothesis to conclude that $T(q, \neg\theta(m, x \oplus \dot{r}_{gen}), k)$ is uniformly $\Pi^{0,x}_n$ in terms of Φ_q, λ_q, ε_q, θ, m, and k. Moreover, saying whether a $\Pi^{0,x}_n$ subtree of a recursively bounded recursive tree is infinite is itself $\Pi^{0,x}_n$, indeed it is $\Sigma^{0,x}_n$ to state that there is a splitting level in the recursive tree which is disjoint from the $\Pi^{0,x}_n$ subtree. Therefore, $\boldsymbol{\tau} \in T(p, \psi, k)$ is a $\Pi^{0,x}_{n+1}$ property of $\boldsymbol{\tau}$, p, ψ and k. □

Construction of the decreasing sequence: We pose $p_0 = (\varnothing, \varnothing, \varepsilon, 1)$ (where ε is the empty string). At stage $s \in \omega$ of the construction we inductively assume that:

- we have $p = p_{s-1} = (\Phi_p, \mathbf{X}_p, \lambda_p, \varepsilon_p)$
- we know the indices (of the computable functions) computing each member of $\mathbf{X}_p$ from $x^{(n+1)}$
- $\mathbf{X}_p$ doesn't include any κ-name of $y \in [0,1]^\omega$

Let $\psi(x \oplus \dot{r}_{gen})$ be the s-th Π^0_{n+1} sentence. In particular, $\psi(x \oplus \dot{r}_{gen})$ is of the form $\forall m \theta(m, x \oplus \dot{r}_{gen})$ with $\theta(m, x \oplus \dot{r}_{gen})$ Σ^0_n formula. We describe how to build the next condition $p_s \leq p_{s-1}$ (that we will call r to avoid too many subscripts) which forces either ψ or $\neg\psi$. Such construction is different depending on the totality of $y \oplus x^{(n+1)}$:
$y \oplus x^{(n+1)}$ is non total: We fix a name α_y of y[12], and divide the construction in two phases:

Phase 1: We force the next sentence (ψ or $\neg\psi$) without extending Φ_G along any name of y (in this way we ensure that Φ_G is consistent along y)

Phase 2: We extend Φ_G along the name α_y by coding the result of the previous phase into $\Phi_G(\alpha_y)$ and λ_G.

[12]From now on, until the end of the proof, we write *name* instead of *κ-name*, because there is no ambiguity.

Phase 1: Suppose that $\mathbf{X}_p = \{\beta_1, \dots, \beta_k\}$, for each $i \in \omega$ we define:

$$q_i := (\Phi_p, \mathbf{X}_p, \lambda_p^\frown i, 2^{-i}\varepsilon_p) \leq p$$

We now check if $\exists i \in \omega$ such that $T(q_i, \psi, k+1)$ is infinite, so we have two cases:

Case 1 (forcing ψ): If such i exists, we prove that $T(q_i, \psi, k+1)$ has an $x^{(n+1)}$-computable infinite path $\mathbf{X}$ which does not include any name of y. Indeed, by the previous lemma $P = [T(q_i, \psi, k+1)]$ is a $\Pi_1^{0,x^{(n)}}$ set in the effectively compact space $[\boldsymbol{H}]^{k+1}$, therefore we can apply the Cone avoidance Lemma 2.45 and obtain $\overline{Y} = (\gamma_0, \dots, \gamma_k) \in P$ such that $y \nleq_M \overline{Y} \leq y \oplus x^{(n+1)}$. In particular, since $y \nleq_M \overline{Y}$ then for every $j \leq k$ we have $\kappa(\gamma_j) \neq y$. Therefore, some open ball separates $\kappa(\gamma_j)$ from y, that is $\exists l_j \in \omega$ such that $y \notin B^*_{\gamma_j \upharpoonright l_j}$. We consider $l = \max_{j \leq k} l_j$ and the subtree T consisting of $(\tau_j)_{j \leq k} \in T(q_i, \psi, k+1)$ such that $\forall j \leq k(\tau_j \trianglelefteq$-comparable with $\gamma_j \upharpoonright l_j)$. Therefore, T is a nonempty $\Pi_1^{0,x^{(n)}}$ subtree of $\boldsymbol{H}^{k+1}$, and it contains a $y \oplus x^{(n+1)}$-computable infinite path $\mathbf{X}$ (which doesn't contain any name of y).
Therefore, using the second point of the Lemma 2.57, we can force ψ considering:

$$q := (\Phi_p, \mathbf{X}_p \cup \mathbf{X}, \lambda_p^\frown i, 2^{-i}\varepsilon_p)$$

Case 2 (forcing $\neg\psi$): If we have that $\forall i \in \omega (T(q_i, \psi, k+1)$ is finite), then $\forall i \in \omega \exists l \in \omega ((\beta_1 \upharpoonright l, \dots, \beta_k \upharpoonright l, \alpha_y \upharpoonright l) \notin T(q_i, \psi, k+1))$, that is:

$$\begin{aligned} &\forall i \in \omega \exists \hat{r}_i \leq q_i^0 \exists m \in \omega (\hat{r}_i \Vdash_{\mathbb{P}_{\mathrm{KS}}} \neg\theta(m, x \oplus \dot{r}_{gen}) \wedge \\ &\quad \forall \beta \in \mathrm{dom}(\Phi_{\hat{r}_i}) \setminus \mathrm{dom}(\Phi_p)(b \text{ does not meet } \mathbf{X}_p \cup \{\alpha_y\}) \wedge \\ &\qquad \sum\{2^{-\ell(\sigma)} \mid \sigma \in \mathrm{dom}(\Phi_q) \setminus \mathrm{dom}(\Phi_p)\} \leq 2^{-i}\varepsilon_p) \end{aligned}$$

We observe that any of this $\hat{r}_i$ force $\neg\psi$ without adding elements that meet α_y. However, their research requires knowing also α_y and hence cannot be performed in $y \oplus x^{(n+1)}$-computably manner (because $y \oplus x^{(n+1)}$ is not total). Thus, we need to replace them with conditions r_i that are easier to find. In particular, thanks to Lemma 2.57, we have that for a given $q \in \mathbb{P}_{\mathrm{KS}}$ are equivalent:

1. $\exists r \leq q \exists m \in \omega (r \Vdash_{\mathbb{P}_{\mathrm{KS}}} \neg\theta(m, x \oplus \dot{r}_{gen}))$
2. $\exists r (r^0 \leq q^0 \wedge \exists m \in \omega \exists l \geq |\mathbf{X}_q| (T(r, \neg\theta(m, x \oplus \dot{r}_{gen}), l)$ is infinite))

Therefore, instead of searching for $\hat{r}_i$ we look for the first triple (r_i, l, m) such that $r_i = (\Phi_{r_i}, \varnothing, \lambda_{r_i}, \varepsilon_{r_i}) \leq q_i^0$, $T(r_i, \neg\theta(m, x \oplus \dot{r}_{gen}), l)$ is infinite, $\forall \beta \in \mathrm{dom}(\Phi_{r_i}) \setminus \mathrm{dom}(\Phi_p)(\beta$ does not meet $\mathbf{X}_p)$ (but can meet α_y) and $\sum\{2^{-\ell(\sigma)} \mid \sigma \in \mathrm{dom}(\Phi_q) \setminus \mathrm{dom}(\Phi_p)\} \leq 2^{-i}\varepsilon_p$. We observe that this search is computable given Φ_p, λ_p, ε_p, $x^{(n+1)}$, and the indices that compute each member of $\mathbf{X}_p$ from $x^{(n+1)}$. Therefore, we only have to ensure that there is some $i_0 \in \omega$ such that $\mathrm{dom}(\Phi_{r_{i_0}}) \smallsetminus \mathrm{dom}(\Phi_p)$ does not meet α_y.

Claim 6. If $y \oplus x^{(n+1)}$ is not total, then $\exists i \in \omega \forall \sigma \in \mathrm{dom}(\Phi_{r_i}) \smallsetminus \mathrm{dom}(\Phi_p)$ (σ does not meet α_y) (that is $B^*_\sigma \cap B^*_{\alpha_y \restriction \ell(\sigma)} = \varnothing$).

Proof of the Claim. Towards a contradiction, assume that $\forall i \in \omega \exists \sigma \in \mathrm{dom}(\Phi_{r_i}) \smallsetminus \mathrm{dom}(\Phi_p)(\sigma$ meet $\alpha_y)$ (that is $B^*_\sigma \cap B^*_{\alpha_y \restriction \ell(\sigma)} \neq \varnothing$). Let B'_σ be the open ball with same center as B^*_σ and radius of $3 \cdot 2^{-\ell(\sigma)}$. By the definition of $\boldsymbol{H}$ the diameter of B^*_τ is less than $2^{-\ell(\tau)}$, therefore $B^*_\sigma \cap B^*_{\alpha_y \restriction \ell(\sigma)} \neq \varnothing$ implies that $y \in B^*_{\alpha_y \restriction \ell(\sigma)} \subseteq B'_\sigma$. Now, consider:

$$y \in E_i \coloneqq \bigcup \{B'_\sigma \mid \sigma \in \mathrm{dom}(\Phi_{r_i}) \smallsetminus \mathrm{dom}(\Phi_p)\}$$

we observe that $\sum \{l \mid l \text{ radius of } B'_\sigma \wedge \sigma \in \mathrm{dom}(\Phi_{r_i}) \smallsetminus \mathrm{dom}(\Phi_p)\} \leq 3 \cdot 2^{-i} \varepsilon_p$ for the choice of the r_is.

Definition 2.59. We say that two balls B'_σ and B'_τ with $\sigma, \tau \in \mathrm{dom}(\Phi_{r_i}) \smallsetminus \mathrm{dom}(\Phi_p)$ are in the **same connected component** if there exists a sequence $\sigma_0, \dots, \sigma_N \in \mathrm{dom}(\Phi_{r_i}) \smallsetminus \mathrm{dom}(\Phi_p)$ such that $B'_\sigma = B'_{\sigma_0}$, $B'_\tau = B'_{\sigma_N}$ and $\forall j < N(B'_{\sigma_j} \cap B'_{\sigma_{j+1}} \neq \varnothing)$.

We enumerate all distinct connected components of E_i as $\{U_{i,j} \mid j < J(i)\}$ in a fixed way. Observe that the maximal distance between any two points in the same connected component is at most $12 \cdot 2^{-i} \varepsilon_p$ (that is smaller than 2^{-i+4} because $\varepsilon_p < 1$ and $12 < 2^4$). Thus, we let $C_{i,j}$ be a ball of radius 2^{-i+4} which contains $U_{i,j}$. We define $z \in \omega^\omega$ in the following way:

$$z(i) = j \Leftrightarrow j \text{ is the unique component such that } y \in U_{i,j}$$

We have that $y \oplus x^{(n+1)} \equiv_M z \oplus x^{(n+1)}$, indeed:

- $y \leq_M z \oplus x^{(n+1)}$ because we can compute Φ_{r_i} and $\{C_{i,j} \mid i \in \omega \wedge j < J(i)\}$ in a $x^{(n+1)}$-computable way (as one can effectively decide whether two basic open balls intersect using $\varnothing'$ as oracle, to compute the different components). Moreover, since $\forall i \in \omega(y \in U_{i,z(i)} \subseteq C_{i,z(i)})$ and $\mathrm{diam}(C_{i,z(i)}) \underset{i \to \infty}{\longrightarrow} 0$, $\{y\} = \bigcap_{i \in \omega} C_{i,z(i)}$.
- $z \leq_M y \oplus x^{(n+1)}$ because, similarly as above, we can compute $\{U_{i,j} \mid i \in \omega \wedge j < J(i)\}$ in a $x^{(n+1)}$-computably manner. In addition, for each $i \in \omega$ and any name of y we can compute the unique $z(i)$ such that $y \in U_{i,z(i)}$ (because the components are pairwise disjoint).

But this means that $y \oplus x^{(n+1)}$ is total, and is against the assumptions. ↯ □

So we consider the $i_0 \in \omega$ such that $\mathrm{dom}(\Phi_{r_{i_0}}) \setminus \mathrm{dom}(\Phi_p)$ does not meet α_y. By the property of the r_is, we have that the corresponding tree $T(r_{i_0}, \neg\theta(m, x \oplus \dot{r}_{gen}), l)$ is infinite for some $m \in \omega$ and $l \geq k$, so we take as next forcing condition:

$$q := (\Phi_{r_{i_0}}, \mathbf{X}_p \cup \mathbf{X}, \lambda_{r_{i_0}}, \varepsilon_{r_{i_0}})$$

where $\mathbf{X}$ is a set of size l of $x^{(n+1)}$-computable elements in $\boldsymbol{H}$ containing no name of y, which is obtained by applying the Cone avoidance Lemma 2.45 to the tree $T(\Phi_{r_{i_0}}, \neg\theta(m, x \oplus \dot{r}_{gen}), l)$ and the steps as in Case 1. It is immediate to check that $q \leq q_i \leq p$.

So the Phase 1 ends producing a new condition $q = (\Phi_q, \mathbf{X}_q, \lambda_q, \varepsilon_q) \leq p_s$.
Phase 2: Suppose that $\mathbf{X}_q = \{\beta_1, \ldots, \beta_m\}$ we consider the least $b \in \omega$ and the least $u \geq s$ such that $\forall i = 1, \ldots, m(B^*_{\alpha_y \restriction u} \cap B^*_{\beta_i \restriction b} = \varnothing)$ (such b and u exist because $\mathbf{X}_q$ does not contain any name of y). In addition, consider:

- an index j_0 which computes the l-tuple corresponding to $\mathbf{X}_q$ with $x^{(n+1)}$ as oracle
- for every $\sigma \in \omega^{<\omega}$, $\sigma^+ \in \omega^{<\omega}$ defined by $\forall i \in \omega(\sigma^+(i) = \sigma(i) + 1)$
- $k = 0$ if $q \Vdash_{\mathbb{P}_{\mathrm{KS}}} \psi(x \oplus \dot{r}_{gen})$ and $k = 1$ if $q \Vdash_{\mathbb{P}_{\mathrm{KS}}} \neg\psi(x \oplus \dot{r}_{gen})$

so we define the next condition as:

$$r = (\Phi_q \cup \{(s, k, \alpha_y \restriction u)\}, \mathbf{X}_q, \lambda_q^\frown j_0^\frown (\alpha_y \restriction u)^{+\frown} 0, \varepsilon_q - 2^{-u})$$

By construction, we have that $r \leq q \leq p$ and $\Phi_r(\alpha_y \restriction u)(s) = 0$ if and only if r forces the s-th Π^0_{n+1} sentence $\psi(x \oplus \dot{r}_{gen})$.
Verification of the thesis: From our construction we obtain a decreasing sequence $(p_s)_{s\in\omega}$ and we define $G := x \oplus \Phi_G \oplus \lambda_G$ with $\Phi_G := \bigcup_{n\in\omega} \Phi_{p_n}$ and $\lambda_G := \bigcup_{n\in\omega} \lambda_{p_n}$. Therefore, $\Phi_G(\alpha_y)$ is the characteristic function of a complete $\Sigma^{0,G}_{n+1}$ set and hence $\Phi_G(\alpha_y) \equiv_T G^{(n+1)}$ (actually, they are many-one equivalent). We have:

Claim 7. Φ_G is consistent along y.

Proof of the Claim. A new computation is added into Φ_G only when we construct r_{i_0} from p in Case 2 of Phase 1, and when we construct r from q in Phase 2. By the fact that $\forall \sigma \in \mathrm{dom}(\Phi_{r_{i_0}}) \setminus \mathrm{dom}(\Phi_p)(\sigma$ does not meet $\alpha_y)$, $\Phi_{r_{i_0}}$ does not add any new computation along y. Moreover, our construction in Phase 2 does not add any inconsistent computation. □

By combining the last and the first claims of this proof, we obtain that $G^{(n+1)} \equiv_T \Phi_G(\alpha_y) \leq_M \Phi_G \oplus y \equiv_M G \oplus y$. Moreover, $x \leq_M G$ implies that $G \oplus x^{(n+1)} \leq_M G^{(n+1)}$. Finally, we have that the whole construction is $G \oplus x^{(n+1)}$-computable.

Claim 8. If $y \oplus x^{(n+1)}$ is not total, then $\alpha_y \leq_M G \oplus x^{(n+1)}$.

Proof of the Claim. Suppose that at stage s of the construction we have computed the finite string representing Φ_p, the index j_p corresponding to the computations of the elements of $\mathbf{X}_p$ (relative to the oracle $x^{(n+1)}$), the finite string λ_p, and the rational ε_p. We decode from λ_G the unique number i_0 such that $\lambda_p^\frown i_0 \prec \lambda_G$. Then, we use $x^{(n+1)}$ to check whether the tree $T(p_{i_0}, \psi, k+1)$ is infinite or not:

1. $T(p_{i_0}, \psi, k+1)$ is infinite: we know that the output from Phase 1 is $(\Phi_p, \mathbf{X}_p \cup \mathbf{X}, \lambda_p^\frown i_0, 2^{i_0}\varepsilon_p)$ for some $\mathbf{X}$, and so we decode j_0 and $\alpha_y \upharpoonright u$ since an initial segment of λ_G is of the form $\lambda_p^\frown i_0^\frown j_0^\frown (\alpha_y \upharpoonright u)^{+\frown} 0$ by the construction in Phase 2. By using $\alpha_y \upharpoonright u$ and Φ_G, we can also decode the value of $z \in \omega^\omega$ of Claim 6 in Phase 1. These codes tell us the full information on the next forcing condition r (and hence allow us to recover it) and from it we continue the computation of α_y.

2. $T(p_{i_0}, \psi, k+1)$ is finite: in this case we know that the output from Phase 1 is $(\Phi_{r_{i_1}}, \mathbf{X}_p \cup \mathbf{X}, \lambda_{r_{i_1}}, \varepsilon_{r_{i_1}})$ for some $i_1 \in \omega$. Moreover, by construction, we have that $r_{i_1} \leq q_{i_1}$ and since $\lambda_{q_{i_1}} = \lambda_p^\frown i_1$, we conclude that $i_0 = i_1$. Thus, we search for r_{i_0} using the oracle $x^{(n+1)}$ and the indices for elements of $\mathbf{X}_p$. The first $\Phi_{r_{i_0}}$ found must avoid α_y by Claim 6 of Phase 1. Having found r_{i_0}, we can proceed as in the infinite case to recover j_0 and $\alpha_y \upharpoonright u$ and full information on the next forcing condition. □

To sum up, if $y \oplus x^{(n+1)}$ is not total, we obtain $G \oplus x^{(n+1)} \equiv_M G^{(n+1)} \equiv_M G \oplus y$ because:

$$G^{(n+1)} \leq_M G \oplus y \underset{\substack{| \\ y \leq_M \alpha_y \leq_M G \oplus x^{(n+1)}}}{\leq_M} G \oplus x^{(n+1)} \leq_M G^{(n+1)}$$

<u>$y \oplus x^{(n+1)}$ is total:</u> In this case, we do not have the property $y \leq_M G \oplus x^{(n+1)}$. Instead, we proceed using a $y \oplus x^{(n+1)}$-computable construction in which the parameters λ_p and ε_p have no role. Being $y \oplus x^{(n+1)}$ total, it computes a canonical name of y, that is, we can choose a name $\alpha_y \in [\boldsymbol{H}]$ of $y \in [0,1]^\omega$ such that $\alpha_y \leq_T y \oplus x^{(n+1)}$ (thanks to Fact 1 point 4.). We briefly explain how to adapt the proof for the non total case to the total one.

Phase 1: We check whether $\exists i \in \omega$ such that $T(q_i, \psi, k+1)$ is infinite or not, so we have two cases:

Case 1 (forcing ψ): As before $P = [T(q_i, \psi, k+1)]$ is a $\Pi_1^{0,x^{(n)}}$ set in $[\boldsymbol{H}]^{k+1}$, therefore we can apply the Cone avoidance Lemma 2.45 and obtain $\overline{Y} \in P$ such that $y \not\leq_M \overline{Y} \leq_M y \oplus x^{(n+1)} \equiv_M \alpha_y \oplus x^{(n+1)}$. Thus, we can proceed as in the non total case and the added $\mathbf{X}$ is $\alpha_y \oplus x^{(n+1)}$-computable.

Case 2 (forcing $\neg\psi$): We skip the construction of r_i and use $\alpha_y \oplus x^{(n+1)}$ to effectively find r_0 (corresponding to the r_{i_0} in the non total case). Finally, we take the next forcing condition to be $(\Phi_{r_0}, \mathbf{X}_p \cup \mathbf{X}, \lambda_{r_0}, \varepsilon_{r_0})$ as in the last paragraph in Case 2 of Phase 1 of non total case.

Phase 2: We repeat the same steps and, moreover, find the parameters u and b using the oracle $\alpha_y \oplus x^{(n+1)}$.

Therefore, the entire construction is $(\alpha_y \oplus x^{(n+1)})$-computable and hence $\Phi_G(\alpha_y) \leq_M \alpha_y \oplus x^{(n+1)} \equiv_M y \oplus x^{(n+1)}$. As before, $\Phi_G(\alpha_y) \equiv_T G^{(n+1)}$ and (by the consistency of Φ_G) $G^{(n+1)} \leq_M G \oplus y$. Moreover, having $x \leq_M G$ we also have that $G \oplus y \equiv_M y \oplus x^{(n+1)}$, because:

$$G^{(n+1)} \leq_M y \oplus x^{(n+1)} \leq_M y \oplus G^{(n+1)} \leq_M G \oplus y \leq G^{(n+1)} \oplus y$$

Finally, as $y \oplus x^{(n+1)}$ total, we use the Friedberg Jump Inversion Theorem 2.47 to $y \oplus x^{(n+1)} \geq_M G^{(n+1)}$ and get $\tilde{G} \geq_T G$ such that $\tilde{G}^{(n+1)} \equiv_T y \oplus x^{(n+1)}$. In particular, $\tilde{G}^{(n+1)} \equiv_M \tilde{G} \oplus y \equiv_M y \oplus x^{(n+1)}$ because:

$$\tilde{G}^{(n+1)} \equiv_T y \oplus x^{(n+1)} \geq_M y \oplus \tilde{G} \geq_M y \oplus G \equiv_M \tilde{G}^{(n+1)}$$

For transfinite cases, the uniformity of this constructions allows to prove the theorem for the limit ordinals as well and for successor ordinals one can conduct an analysis on the complexity of the forcing relation similar to lemmas 2.57 and 2.58. Therefore, we can lift the result to all the recursive ordinals. □

We notice that even this "non-uniform" version of Shore-Slaman Join Theorem for continuous degrees implies the Posner-Robinson Theorem.

Theorem 2.60 (Posner-Robinson for recursive spaces)**.** Given $\mathcal{X}$ and $\mathcal{Y}$ recursive spaces, $\forall x \in X \forall y \in Y (y \leq_M x \dot\vee \exists g \in 2^\omega (x \oplus y \oplus g \geq_M (g \oplus x)'))$.

Proof. Given $x \in X$ and $y \in Y$ either $y \leq_M x$ or $y \not\leq_M x$. Suppose to be in the second case, and consider the $g = G \in 2^\omega$ given by the Shore-Slaman Join Theorem, hence: $g \geq_M x \Rightarrow g \oplus x \equiv_M g$. Therefore:

$$(g \oplus x)' \equiv_M g' \equiv_M y \oplus g \equiv_M y \oplus g \oplus x$$ □

Chapter 3

Applications of the Shore-Slaman Join Theorem

In this chapter, we exhibit two applications of the Shore-Slaman Join Theorem to obtain decomposability results in (Effective) Descriptive Set Theory. The first section presents a decomposability result for Borel functions, corresponding to [GKN21, Theorem 1.1]. In the final section, we present a weak form of the Solecki Dichotomy for Borel functions between Polish spaces. This original contribution builds on a game-theoretic argument initially presented in [Lut23].

3.1 A decomposability result for Borel functions

Together with the decomposability result presented in [GKN21, Theorem 1.1] we briefly describe in this section the context that motivated it.

3.1.1 The Borel Decomposability Conjecture in Descriptive Set Theory

Definition 3.1. Given $\mathbf{\Gamma}_0$ and $\mathbf{\Gamma}_1$ boldface pointclasses and a function $f : X \to Y$ between topological spaces, we write $f^{-1}\mathbf{\Gamma}_0 \subseteq \mathbf{\Gamma}_1$ if $f^{-1}[U] \in \mathbf{\Gamma}_1(X)$ for any set $U \in \mathbf{\Gamma}_0(Y)$.

Clearly, this definition can be stated in a similar way for lightface pointclasses and basic spaces.
Jayne and Rogers in [JR82] established the following characterization:

Theorem 3.2 (Jayne-Rogers Theorem [JR82])**.** Given X analytic subset of a Polish space, Y separable metrizable space, and a function $f : X \to Y$, then $f^{-1}\,\mathbf{\Sigma}^0_2 \subseteq \mathbf{\Sigma}^0_2$ if and only if it is countable union of continuous functions with closed domains, i.e.

$$f = \bigcup_{n\in\omega} f_n \text{ where } f_n : \mathrm{dom}(f_n) \to Y \text{ continuous and } \mathrm{dom}(f_n) \in \mathbf{\Pi}^0_1(X)$$

In light of this result, we introduce the following notation:

Notation 3.3. Given X, Y topological spaces, $f : X \to Y$, and $\mathbf{\Gamma}$, $\mathbf{\Delta}$ boldface pointclasses we write:

- $f \in \mathrm{dec}(\mathbf{\Gamma})$ if there exists a partition $\{A_n\}_{n\in\omega}$ of X such that $f \upharpoonright A_n$ is $\mathbf{\Gamma}$-measurable for all $n \in \omega$.
- $f \in \mathrm{dec}(\mathbf{\Gamma}, \mathbf{\Delta})$ if $f \in \mathrm{dec}(\mathbf{\Gamma})$ and moreover the partition is made of $\mathbf{\Delta}$ sets (that is $\{A_n\}_{n\in\omega} \subseteq \mathbf{\Delta}(X)$).

Following this notation the Jayne-Rogers Theorem 3.2 can be restated as: $f^{-1}\,\mathbf{\Sigma}^0_2 \subseteq \mathbf{\Sigma}^0_2 \Leftrightarrow f \in \mathrm{dec}(\mathbf{\Sigma}^0_1, \mathbf{\Delta}^0_2)$. Indeed:

Remark 3.4. Given X, Y separable metrizable spaces, $f : X \to Y$, and $\mathbf{\Gamma}$ boldface pointclass, then $f \in \mathrm{dec}(\mathbf{\Gamma}, \mathbf{\Delta}^0_n)$ if and only if

$$f \text{ is countable union of } \mathbf{\Gamma}\text{-measurable functions with domains in } \mathbf{\Pi}^0_{n-1}(X)$$

Notice that, in the latter term of the equivalence, the domains on which the restrictions of f are $\mathbf{\Gamma}$-measurable are not necessarily disjoint.

The Jayne-Rogers Theorem sparked a lot of attention among researchers. In particular, it has lead to the following:

Conjecture (Decomposability Conjecture [GKN21])**.** Given X analytic subset of a Polish space and Y separable metrizable space[1], then for every function $f : X \to Y$ and for every $1 \le m \le n$:

$$f^{-1}\,\mathbf{\Sigma}^0_m \subseteq \mathbf{\Sigma}^0_n \Leftrightarrow f \in \mathrm{dec}(\mathbf{\Sigma}^0_{n-m+1}, \mathbf{\Delta}^0_n)$$

We observe that the right-to-left implication is always satisfied, indeed: given $f \in \mathrm{dec}(\mathbf{\Sigma}^0_{n-m+1}, \mathbf{\Delta}^0_n)$ with witnesses domains $\{A_n\}_{n\in\omega} \subseteq \mathbf{\Delta}^0_n(X)$ and $U \in \mathbf{\Sigma}^0_m(Y)$ then

$$f^{-1}[U] = \bigcup_{j\in\omega} \underbrace{(f \upharpoonright A_j)^{-1}[U]}_{\in \mathbf{\Sigma}^0_n(A_j)}$$

[1] Since one can always replace Y with its completion, without loss of generality one can assume that Y is Polish.

and hence $f^{-1}[U] \in \mathbf{\Sigma}^0_n(X)$ because countable union of intersections of a $\mathbf{\Sigma}^0_n$ with a $\mathbf{\Delta}^0_n$ (the domain A_n).
Gregoriades, Kihara and Ng in [GKN21], developed the theory of continuous degrees in recursively presented metric spaces to prove the following decomposability result in the other direction:

Theorem 3.5 ([GKN21, Theorem 1.1])**.** Given X, Y Polish spaces, an analytic subset $A \in \mathbf{\Sigma}^1_1(X)$, $n \geq m \geq 1$, and $f : A \to Y$ then:

$$f^{-1}\,\mathbf{\Sigma}^0_m \subseteq \mathbf{\Sigma}^0_n \Rightarrow f \in \operatorname{dec}(\mathbf{\Sigma}^0_{n-m+1})$$

Moreover, under additional hypothesis on the function they show how to obtain the desired complexity of the domains.

Corollary 3.6 ([GKN21, Theorem 1.1])**.** Given X, Y Polish spaces, an analytic subset $A \in \mathbf{\Sigma}^1_1(X)$, $n \geq m \geq 3$, and $f : A \to Y$ then:

1. $f^{-1}\,\mathbf{\Sigma}^0_m \subseteq \mathbf{\Sigma}^0_n \wedge f$ $\mathbf{\Sigma}^0_{n-1}$-measurable $\Rightarrow f \in \operatorname{dec}(\mathbf{\Sigma}^0_{n-m+1}, \mathbf{\Delta}^0_n)$
2. $f^{-1}\,\mathbf{\Sigma}^0_m \subseteq \mathbf{\Sigma}^0_n \wedge \operatorname{graph}(f) \in \mathbf{\Sigma}^0_m(X \times Y) \Rightarrow f \in \operatorname{dec}(\mathbf{\Sigma}^0_{n-m+1}, \mathbf{\Delta}^0_n)$

Observe that their result does not imply the Jayne-Rogers Theorem.

3.1.2 The decomposability result

Before presenting the proof of Theorem 3.5, we need to introduce a crucial definition (see Theorem 3.13 and Lemma 3.14).

Definition 3.7 ([GKN21, Definition 1.3])**.** Given Γ_0, Γ_1 pointclasses (lightface or boldface) and $\mathbf{\Lambda}$ boldface pointclass, a parametrization system for Γ_0 and Γ_1 said $(E^Z_{\Gamma_0})_Z$ and $(E^Z_{\Gamma_1})_Z$ (respectively) and a function $f : X \to Y$ (between basic or topological spaces) we say that $f^{-1}\Gamma_0 \subseteq \Gamma_1$ **holds $\mathbf{\Lambda}$-uniformly** (with respect to $E^Y_{\Gamma_0}$, $E^X_{\Gamma_1}$) if $f^{-1}\Gamma_0 \subseteq \Gamma_1$ holds and, in addition, there exists a $\mathbf{\Lambda}$-measurable function $u : \omega^\omega \to \omega^\omega$ such that:

$$\forall \alpha \in \omega^\omega \forall x \in X (E^Y_{\Gamma_0}(\alpha, f(x)) \Leftrightarrow E^X_{\Gamma_1}(u(\alpha), x))$$

Similarly, if such function $u : \omega^\omega \to \omega^\omega$ is Σ^0_1-recursive we say that $f^{-1}\Gamma_0 \subseteq \Gamma_1$ **holds recursive-uniformly** (or Σ^0_1-uniformly).

Uniformity conditions usually are stated for Σ^0_1-recursive transitions in the codes (see for example [Mos09, Section 3H]) because these are the ones that occur more often. We give here some examples (similar to [Lou19, Remark 3.4.6.]).

Proposition 3.8. Given $\mathcal{X}$ and $\mathcal{Y}$ recursive spaces, then:

$$f : \mathcal{X} \to \mathcal{Y} \ \Sigma^0_1\text{-recursive} \Rightarrow f^{-1}\Sigma^0_1 \subseteq \Sigma^0_1 \ \Sigma^0_1\text{-uniformly w.r.t. } W^{\Sigma^0_1,\mathcal{Y}}, W^{\Sigma^0_1,\mathcal{X}}$$

Proof. We define $D = \{(n,x) \in \omega \times X \mid f(x) \in W^{\Sigma^0_1,\mathcal{X}}_n\}$, it is Σ^0_1 because is defined by recursive substitution. Hence, by Theorem 1.41, there is a recursive $f_D : \omega \to \omega$ such that:

$$D(n,x) \Leftrightarrow W^{\Sigma^0_1,\mathcal{X}}(f_D(n),x)$$

thus, the thesis follows. □

Actually, using the S-m-n Theorem we have already showed in Lemma 2.40 that the same result holds for the parametrization system $(G^{(\mathcal{Y},\varepsilon)}_{\Sigma^0_1})_{(\mathcal{Y},\varepsilon)}$.

Proposition 3.9. Given $\mathcal{X}$ and $\mathcal{Y}$ recursive spaces, then:

$$f : \mathcal{X} \to \mathcal{Y} \ \Sigma^0_1\text{-recursive} \Rightarrow f^{-1}\Sigma^0_1 \subseteq \Sigma^0_1 \ \Sigma^0_1\text{-uniformly w.r.t. } G^{\mathcal{Y}}_{\Sigma^0_1}, G^{\mathcal{X}}_{\Sigma^0_1}$$

Proof. That $f^{-1}\Sigma^0_1 \subseteq \Sigma^0_1$ holds follows from Proposition 1.10. Moreover, by slightly modifying the proof in the claim of Lemma 2.40 we get that $f^{-1}\Sigma^{0,\alpha}_1 \subseteq \Sigma^{0,\alpha}_1$ holds recursive-uniformly for any the oracle α, thus the thesis follows. We explicitly write here the proof for completeness: f Σ^0_1-recursive means that for some semirecursive $D^* \subseteq \omega^2$

$$(x,n) \in D_f \Leftrightarrow \exists m \in \omega (x \in V^{\mathcal{X}}_m \wedge D^*(n,m))$$

therefore

$$\begin{aligned} H^{\mathcal{Y}}_{\Sigma^0_1}(\alpha, e, f(x)) &\Leftrightarrow \exists n \in W^\alpha_e (f(x) \in V^{\mathcal{Y}}_n) \\ &\Leftrightarrow \exists n, m \in \omega (n \in W^\alpha_e \wedge D^*(n,m) \wedge x \in V^{\mathcal{X}}_m) \end{aligned}$$

we now define:

$$h(e,m) = \begin{cases} 1 & \text{if } \exists n \in \omega (n \in W^\alpha_e \wedge D^*(n,m)) \\ \uparrow & \text{otherwise} \end{cases}$$

again, such function is α-computable because it graph is $\Sigma^{0,\alpha}_1$, hence $\varphi^\alpha_j = h$ for some $j \in \omega$, and by the S-m-n Theorem there is an injective recursive function S such that: $\varphi^\alpha_j(e,m) = \varphi^\alpha_{S(j,e)}(m)$. Hence, by construction:

$$H^{\mathcal{X}}_{\Sigma^0_1}(\alpha, S(j,e), x) \Leftrightarrow \exists m \in W^\alpha_{S(j,e)} (x \in V^{\mathcal{X}}_m) \Leftrightarrow H^{\mathcal{Y}}_{\Sigma^0_1}(\alpha, e, f(x))$$ □

To prove Theorem 3.5, we need deep results from Effective Descriptive Set Theory (together to the Shore-Slaman Join Theorem). In particular, we need to recall that

Definition 3.10. Given $\mathcal{X}$ recursive space and a lightface pointclass Γ, a point $x \in X$ is **Γ-recursive** if $\mathrm{N}_{\text{base}}(x) = \{n \in \omega \mid x \in V_n^{\mathcal{X}}\} \in \Gamma(\omega)$. In this case we write $x \in \Gamma$.

Theorem 3.11 (Π^1_1-uniformization Theorem [Lou19, Theorem 5.1.6])**.** Given $\mathcal{X}$ recursive space and $\mathcal{Y}$ recursively presented Polish space, and $P \in \Pi_1^{1,\varepsilon}(\mathcal{X} \times \mathcal{Y})$ for some $\varepsilon \in \omega^\omega$, such that $\forall x \in X \exists y \in \Delta_1^{1,(\varepsilon,x)}(P(x,y))$. Then there exists a $\Delta_1^{1,\varepsilon}$-recursive function $f : \mathcal{X} \to \mathcal{Y}$ which uniformizes P (i.e. such that $\forall x \in X(P(x, f(x))))$.

The Π^1_1-uniformization Theorem is stated in [Lou19] for $\mathcal{Y}$ co-Souslin space (that is spaces such that $\delta_{\mathcal{Y}}[Y]$ is Π^1_1 —where $\delta_{\mathcal{Y}}$ is the function introduced in Example 2.19). However, we state it for $\mathcal{Y}$ recursively presented Polish space because it is sufficient for what we need and, moreover, every recursively presented Polish space is co-Souslin (because $\delta_{\mathcal{Y}}[Y]$ is Π^0_2 —see [Lou19, Proposition 3.5.8.]). We recall that given three sets A, B and C, A is *separated from B by C* if $A \subseteq C$ and $C \cap B = \varnothing$.

Theorem 3.12 (Louveau Separation Theorem [Lou19, Theorem 7.1.10])**.** Given $\mathcal{X}$ Polish recursive space and $A, B \in \Sigma^1_1(\mathcal{X})$ disjoint sets. If A is separated from B by a $\boldsymbol{\Pi}^0_\xi$ set in $\mathcal{X}$, then there is a $\gamma \in \Delta^1_1(\omega^\omega)$ such that A is also separated from B by a $\Pi_\xi^{0,\gamma}$ set.

In particular, these results are needed to prove the following:

Theorem 3.13 (Borel-uniform transition in the codes)**.** Given X and Y Polish spaces, $A \in \boldsymbol{\Sigma}^1_1(X)$ analytic, $f : A \to Y$ and $m, n \geq 1$ then:

$$f^{-1}\,\boldsymbol{\Sigma}^0_m \subseteq \boldsymbol{\Sigma}^0_n \Rightarrow f^{-1}\,\boldsymbol{\Sigma}^0_m \subseteq \boldsymbol{\Sigma}^0_n \text{ holds Borel-uniformly w.r.t. } G^Y_{\Sigma^0_m}, G^X_{\Sigma^0_n}$$

Proof. Without loss of generality, we can consider a sufficiently powerful oracle $\varepsilon \in \omega^\omega$ such that $\mathcal{X}$ and $\mathcal{Y}$ are ε-recursively presented Polish, $A \in \Sigma_1^{1,\varepsilon}$ and f is $\Delta_1^{1,\varepsilon}$-recursive. We define the relations $P, Q \subseteq \omega^\omega \times \mathcal{X}$ as:

$$P(\alpha, x) \Leftrightarrow x \in A \wedge G^{\mathcal{Y}}_{\Sigma^0_m}(\alpha, f(x))$$
$$Q(\alpha, x) \Leftrightarrow x \in A \wedge \neg G^{\mathcal{Y}}_{\Sigma^0_m}(\alpha, f(x))$$

we observe that:

- $G^{\mathcal{Y}}_{\Sigma^0_m} \in \Sigma_m^{0,\varepsilon}(\omega^\omega \times \mathcal{Y})$ and hence $P, Q \in \Sigma_1^{1,\varepsilon}(\omega^\omega \times \mathcal{Y})$.

- For any $\alpha \in \omega^\omega$, the sections $P_\alpha, Q_\alpha \in \Sigma_1^{1,\varepsilon}(\mathcal{Y})$ are disjoint.

We consider $D_\alpha := G^{\mathcal{Y}}_{\Sigma^0_m,\alpha} \in \Sigma_m^{0,\varepsilon\oplus\alpha}(\mathcal{Y})$, then, for our hypothesis on f, we have that $R_\alpha = f^{-1}[D_\alpha] \in \mathbf{\Sigma}^0_n(X)$ and $P_\alpha = R_\alpha \cap A$.
Moreover, since $Q_\alpha \subseteq A$ we also have that R_α separates P_α from Q_α, that is: $P_\alpha \subseteq R_\alpha$ and $Q_\alpha \cap R_\alpha = \varnothing$. Therefore, we can apply the Louveau Separation Theorem 3.12 relativized to ε, and find $S_\alpha \in \Sigma_n^{0,\varepsilon\oplus\gamma}(\mathcal{X})$ that separates P_α from Q_α with $\gamma \in \Delta_1^{1,(\varepsilon,\alpha)}$. In particular, $S_\alpha = G^{\mathcal{X}}_{\Sigma^0_n,\beta}$ for some $\beta \in \omega^\omega$ $\varepsilon\oplus\gamma$-recursive (and hence $\beta \in \Delta_1^{1,(\varepsilon,\alpha)}$).
We define $U \in \omega^\omega \times \omega^\omega$ as the predicates that indicates that $G^{\mathcal{X}}_{\Sigma^0_n,\beta}$ separates P_α from Q_α, that is:

$$\begin{aligned} U(\alpha,\beta) &\Leftrightarrow P_\alpha \subseteq G^{\mathcal{X}}_{\Sigma^0_n,\beta} \wedge Q_\alpha \cap G^{\mathcal{X}}_{\Sigma^0_n,\beta} = \varnothing \\ &\Leftrightarrow \forall x \in P_\alpha (x \in G^{\mathcal{X}}_{\Sigma^0_n,\beta}) \wedge \forall x \in Q_\alpha (x \notin G^{\mathcal{X}}_{\Sigma^0_n,\beta}) \end{aligned}$$

It is clear by the last definition that $U \in \Pi_1^{1,\varepsilon}(\omega^\omega \times \omega^\omega)$, and by the previous part of the proof we have that $\forall \alpha \in \omega^\omega \exists \beta \in \Delta_1^{1,(\varepsilon,\alpha)}(U(\alpha,\beta))$. Therefore, by the Π^1_1-uniformization Theorem 3.11 there exists a $\Delta_1^{1,\varepsilon}$-recursive function (and hence Borel) $u : \omega^\omega \to \omega^\omega$ such that $\forall \alpha \in \omega^\omega (U(\alpha, u(\alpha)))$. This function satisfies the Borel-uniformity in the codes, indeed:

$$\forall \alpha \in \omega^\omega \forall x \in A (G^{\mathcal{Y}}_{\Sigma^0_m}(\alpha, f(x)) \Leftrightarrow x \in P_\alpha \Leftrightarrow x \in G^{\mathcal{Y}}_{\Sigma^0_m}(u(\alpha), x))$$

where the last equivalence holds because $P_\alpha = A \setminus Q_\alpha$. □

Using the Borel-uniform transition, we finally establish a connection with continuous degrees, in particular:

Lemma 3.14. Given $\mathcal{X}$, $\mathcal{Y}$ recursive spaces, and $A \in \Sigma^1_1(\mathcal{X})$. Suppose that a function $f : A \to Y$ satisfies $f^{-1}\,\mathbf{\Sigma}^0_{m+1} \subseteq \mathbf{\Sigma}^0_{n+1}$ Borel-uniformly in the codes $G^Y_{\Sigma^0_m}$, $G^X_{\Sigma^0_n}$ then:

$$\exists z \in 2^\omega \exists \xi \le \omega_1^z \forall q \ge_T z \forall x \in X ((f(x) \oplus q)^{(m)} \le_M (x \oplus q^{(\xi)})^{(n)})$$

Proof. We can consider a sufficiently powerful oracle $\varepsilon \in \omega^\omega$ such that A and Y are ε-computably isomorphic to subspaces of ε-recursively presented Polish spaces $\mathcal{V}$ and $\mathcal{W}$ (respectively). Without loss of generality, we can consider f as function from A into $\mathcal{W}$, indeed every set $S \in \mathbf{\Sigma}^0_{n+1}(Y)$ is of the form $S = \tilde{S} \cap W$ for some $\tilde{S} \in \mathbf{\Sigma}^0_{n+1}(W)$. Therefore we can apply the previous Theorem, and hence $f^{-1}\,\mathbf{\Sigma}^0_{m+1} \subseteq \mathbf{\Sigma}^0_{n+1}$ holds Borel-uniformly. In particular, we also have that: $f^{-1}\,\mathbf{\Sigma}^0_m \subseteq \mathbf{\Sigma}^0_{n+1}$ and $f^{-1}\,\mathbf{\Pi}^0_m \subseteq \mathbf{\Sigma}^0_{n+1}$ hold Borel-uniformly. Thus, there are two Borel functions $u, v : \omega^\omega \to \omega^\omega$ such that:

$$\forall x \in A \forall i^\frown p \in \omega^\omega (f(x) \in G^{\mathcal{W}}_{\Sigma^0_m, i^\frown p} \Leftrightarrow x \in G^{\mathcal{V}}_{\Sigma^0_{n+1}, u(i^\frown p)} \Leftrightarrow x \notin G^{\mathcal{V}}_{\Sigma^0_{n+1}, v(i^\frown p)})$$

Since u and v are Borel, there is some $z \geq_T \varepsilon$ such that u and v are $\Sigma^{0,z}_a$-recursive for some $a \in \mathcal{O}^z$ (where $\mathcal{O}^z$ is the Kleene's $\mathcal{O}$ relative to $z \in 2^\omega$). We observe that for any $p \in 2^\omega$

$$J^{(m),p\oplus z}_{\mathcal{W}}(f(x)) = \{e \in \omega \mid f(x) \in G^{\mathcal{W}}_{\Sigma^0_m, e^\frown p\oplus z}\} = \{e \in \omega \mid x \in G^{\mathcal{V}}_{\Sigma^0_{n+1}, u(w^\frown p\oplus z)}\}$$
$$= \{e \in \omega \mid x \notin G^{\mathcal{V}}_{\Sigma^0_{n+1}, v(i^\frown p\oplus z)}\}$$

is in $\Delta^{0,(p\oplus z)^\xi}_{n+1}(2^\omega)$, where ξ is the countable ordinal coded by a. Hence:

$$\forall p \in 2^\omega (J^{(m),p\oplus z}_{\mathcal{W}}(f(x)) \leq_T J^{(n),(p\oplus z)^{(\xi)}}_{\mathcal{V}}(x))$$

that, by Lemma 2.39, is equivalent to:

$$\forall p \in 2^\omega ((f(x) \oplus p \oplus z)^{(m)} \leq_T (x \oplus (p \oplus z)^{(\xi)})^{(n)})$$

since $p, z \in 2^\omega$, the thesis follows. □

Lemma 3.15 (The Cancellation Lemma)**.** Given $\mathcal{X}$, $\mathcal{Y}$ recursive spaces, $x \in X$ and $y \in Y$ then

$$\forall z \in 2^\omega \forall \xi \leq \omega^z_1 (\forall g \geq_T z((y \oplus g)^m \leq_T (x \oplus g^{(\xi)})^{(n)}) \Rightarrow y \leq_M (x \oplus z^{(\xi)})^{(n-m)})$$

Proof. Towards a contradiction, suppose that $y \nleq_M (x \oplus z^{(\xi)})^{(n-m)}$. We prove

Claim 9. $\exists p \in 2^\omega (p \geq_T z \wedge y \nleq_M p^{(\xi+n-m)} \wedge x \leq_M p^{(\xi)})$

Proof of the Claim. Since $z^{(\xi)}$ is total, by the almost totality (Lemma 2.35):

$$z^{(\xi)} < x \,\dot\vee\, x \oplus z^{(\xi)} \text{ total}$$

Hence, we have two cases:

- <u>$x \oplus z^{(\xi)}$ total:</u> we can apply the ξ-th Friedberg Jump Inversion Theorem 2.47 (on 2^ω), therefore there exists $p \in 2^\omega$ such that $p \geq_T z$ and $p^{(\xi)} \equiv_T x \oplus z^{(\xi)}$, moreover this satisfies the thesis of the claim.
- <u>$z^{(\xi)} < x$:</u> In this case we have $y \nleq_M x^{(n-m)}$ and again we consider two subcases:
  - $m = n$: thus, by Corollary 2.34, exists $\hat{x} \in 2^\omega$ such that $x \leq_M \hat{x}$ and $y \nleq_M \hat{x}$.
  - $m < n$: we consider any $\hat{x} \in \omega^\omega$ such that $\rho_{\mathcal{X}}(\hat{x}) = x$ and $\hat{x}' \equiv_T J^{(1),\varnothing}_{\mathcal{X}}(x)$ (this is possible thanks to Lemma 2.41). Moreover, since $n - m \geq 1$, by Lemma 2.39, $x^{(n-m)} \equiv_T J^{(n-m-1)} \circ J^{(1),\varnothing}_{\mathcal{X}}(x)$ and hence: $\hat{x}^{n-m} \equiv_T x^{(n-m)}$. In particular, it follows that $y \nleq_M \hat{x}^{(n-m)}$.

We observe that in both cases we get a total $\hat{x}$ (and hence, without loss of generality, we can assume that $\hat{x} \in 2^\omega$) such that $x \leq_M \hat{x}$ and $y \nleq_M \hat{x}^{(n-m)}$. Therefore, again by the ξ-th Friedberg Jump Inversion Theorem 2.47, exists $p \in 2^\omega$ such that $p \geq_T z$ and $p^{(\xi)} \equiv_T \hat{x} \oplus z^{(\xi)} \equiv_T \hat{x}$ (because $\hat{x} \geq_M x \geq_M z^{(\xi)}$). Thus $y \nleq_M x^{(n-m)} \equiv_T p^{(\xi+n-m)}$ and $x \leq_M \hat{x} \leq_T p^{(\xi)}$. □

Therefore, by the Shore-Slaman Theorem 2.46, exists $g \in 2^\omega$ such that $g \geq_M p$ and $g^{(n-m+1+\xi)} \equiv_M g \oplus y$, hence:

$$g^{(n+1+\xi)} \equiv_M (g^{(n-m+1+\xi)})^{(m)} \leq_M (g \oplus y)^{(m)}$$

moreover, $g \geq_M p \geq_T z$ implies that $g^{(\xi)} \geq_T p^{(\xi)} \geq_M x$ and hence:

$$(y \oplus g)^m \geq_M g^{(n+1+\xi)} >_T g^{(n+\xi)} \geq_T (x \oplus g^{(\xi)})^{(n)}$$

and this contradicts our assumptions. ↯ □

With these results we are ready to conclude the proof for Theorem 3.5.

Proof of Theorem 3.5. Without loss of generality (considering a sufficiently powerful oracle) we can assume that $\mathcal{X}$ and $\mathcal{Y}$ are recursively presented Polish spaces and $A \in \Sigma^1_1$. We have that $f^{-1}\,\mathbf{\Sigma}^0_m \subseteq \mathbf{\Sigma}^0_n$ and hence $f^{-1}\,\mathbf{\Sigma}^0_{m+1} \subseteq \mathbf{\Sigma}^0_{n+1}$. In particular, by Theorem 3.13 we have that $f^{-1}\,\mathbf{\Sigma}^0_{m+1} \subseteq \mathbf{\Sigma}^0_{n+1}$ holds Borel-uniformly in the codes. Therefore by Lemma 3.14:

$$\exists z \in 2^\omega \exists \xi \leq \omega_1^z \forall q \geq_T z \forall x \in A((f(x) \oplus q)^{(m)} \leq_M (x \oplus q^{(\xi)})^{(n)})$$

and by the Cancellation Lemma:

$$\exists z \in 2^\omega \exists \xi \leq \omega_1^z \forall x \in A(f(x) \leq_M (x \oplus z^{(\xi)})^{(n-m)})$$

For each $e \in \omega$ we define:

$$B_e = \{x \in A \mid f(x) = \Phi_e^{2^\omega,\mathcal{Y}}((x \oplus z^{(\xi)})^{(n-m)})\}^{[2]}$$

we notice that $A = \bigcup_{e\in\omega} B_e$ and $f \upharpoonright B_e$ is $\mathbf{\Sigma}^0_{n-m+1}$-measurable. In fact, each function

$$\begin{aligned} g_e &: A \rightharpoonup \mathcal{Y} \\ &x \mapsto \Phi_e^{2^\omega,\mathcal{Y}}((x \oplus z^{(\xi)})^{(n-m)}) \end{aligned}$$

[2] To be precise, we should write $\Phi_e^{\mathcal{X}\times\omega^\omega,\mathcal{Y}}$ if $n = m$.

is $\Sigma^{0,z^{(\xi)}}_{n-m+1}$-recursive on its domain. Because it is composition of a Σ^0_1-recursive function on its domain $\Phi^{2^\omega,\mathcal{Y}}_e : 2^\omega \rightharpoonup \mathcal{Y}$ and a $\Sigma^{0,z^{(\xi)}}_{n-m+1}$-recursive (total) function, indeed if $n > m$:

$$\begin{aligned}(x \oplus z^{(\xi)})^{(n-m)} \in V^{2^\omega}_k \Leftrightarrow (x \oplus z^{(\xi)})^{(n-m)} \in \prod_{i<\ell(s_k)} \{s_k(i)\} \times \prod_{i\geq \ell(s_k)} 2 \\ \Leftrightarrow \forall i < \ell(s_k)\big((H_{\Sigma^0_{n-m}}(z^{(\xi)}, i, x) \wedge s_k(i) = 1)\vee \\ (\neg H_{\Sigma^0_{n-m}}(z^{(\xi)}, i, x) \wedge s_k(i) = 0)\big)\end{aligned}$$

While for $n = m$, it is immediate. Therefore, $f \in \operatorname{dec}(\mathbf{\Sigma}^0_{n-m+1})$. □

Lemma 3.16. Given X, Y Polish spaces, $A \subseteq X$ and $f : A \to Y$ $\mathbf{\Sigma}^0_\xi$-measurable then:

$$f \in \operatorname{dec}(\mathbf{\Sigma}^0_k) \Rightarrow f \in \operatorname{dec}(\mathbf{\Sigma}^0_k, \mathbf{\Delta}^0_{\max\{\xi,k+1\}+1})$$

Proof. Let $\{A_i\}_{i\in\omega}$ be the partition of A such that $g_i := f \restriction A_i$ is $\mathbf{\Sigma}^0_k$-measurable. We observe that:

Claim 10. Given $B \subseteq X$ and $g : B \to Y$ $\mathbf{\Sigma}^0_\alpha$-measurable, there exist a set $\tilde{B} \in \mathbf{\Pi}^0_{\alpha+1}(X)$ containing B and a function $\tilde{g} : \tilde{B} \to Y$ $\mathbf{\Sigma}^0_\alpha$-measurable which extends g.

Proof of the Claim. Let τ_X be the topology of X and $\{U^Y_n \mid n \in \omega\}$ be a countable basis for Y. We consider the sets $B_{n,i} \in \mathbf{\Delta}^0_\alpha(X)$ such that for each $n \in \omega$

$$f^{-1}[U_n] = B \cap \bigcup_{i\in\omega} B_{n,i}$$

Thanks to [Kec95, Theorem 22.18], there is a Polish topology τ'_X such that $\tau_X \subseteq \tau'_X \subseteq \mathbf{\Sigma}^0_\alpha(X, \tau_X)$ and $\forall i, n \in \omega(B_{n,i} \in \mathbf{\Delta}^0_1(X, \tau'_X))$. Thus, $g : (X, \tau'_X) \rightharpoonup Y$ is partial continuous. Applying [Kec95, Theorem 3.8], we found a $\tilde{B} \in \mathbf{\Pi}^0_2(X, \tau'_X)$ such that $B \subseteq \tilde{B}$ and $\tilde{g} : \tilde{B} \to Y$ continuous w.r.t. the subspace topology induced by (X, τ'_X) which extends g. Thus, w.r.t. the original topology τ_X, $\tilde{B} \in \mathbf{\Pi}^0_{\alpha+1}(X, \tau_X)$ and $g : \tilde{B} \to Y$ is $\mathbf{\Sigma}^0_\alpha$-measurable. □

Therefore, without loss of generality, we can extend each g_i to a $\mathbf{\Sigma}^0_k$-measurable function $\tilde{g}_i : \tilde{A}_i \to Y$ with domain $\tilde{A}_i \in \mathbf{\Pi}^0_{k+1}(X)$. Now, consider the sets

$$B_i = \{x \in \tilde{A}_i \mid f(x) = \tilde{g}_i(x)\} \subseteq A$$

then $f \restriction B_i$ is $\mathbf{\Sigma}^0_k$-measurable and $B_i = (f, \tilde{g}_i)^{-1}[\Delta_Y]$ where $\Delta_Y = \{(y, y) \mid y \in Y\}$ is the diagonal set of Y (which is $\mathbf{\Pi}^0_1$ because Y is Hausdorff) and

$$\begin{aligned}(f, \tilde{g}_i) : \tilde{A}_i &\to Y \times Y \\ x &\mapsto (f(x), \tilde{g}_i(x))\end{aligned}$$

In particular, $B_i = (f, \tilde{g}_i)^{-1}[\Delta_Y] \in \mathbf{\Pi}^0_{\max\{\xi,k+1\}}(A)$ because f is $\mathbf{\Sigma}^0_\xi$-measurable and $\tilde{g}_i$ is $\mathbf{\Sigma}^0_k$-measurable with $\mathrm{dom}(\tilde{g}_i) \in \mathbf{\Pi}^0_{k+1}(X)$. □

In particular, using the previous lemma, we obtain Theorem 3.5 as stated in [GKN21]:

Corollary 3.17. Given X, Y Polish spaces, an analytic subset $A \in \mathbf{\Sigma}^1_1(X)$, $n \geq m \geq 1$, and $f : A \to Y$ then:

$$f^{-1}\,\mathbf{\Sigma}^0_m \subseteq \mathbf{\Sigma}^0_n \Rightarrow f \in \mathrm{dec}(\mathbf{\Sigma}^0_{n-m+1}, \mathbf{\Delta}^0_{n+1})$$

Moreover, in a similar way, we prove Corollary 3.6:

Proof of Corollary 3.6.

1. It follows from Theorem 3.5 and Lemma 3.16, indeed for $n \geq m \geq 3$:

$$\max\{n-1, n-m+2\} + 1 = n$$

2. We already know that: $f \in \mathrm{dec}(\mathbf{\Sigma}^0_{n-m+1})$. Let $\{A_i\}_{i\in\omega}$ be the partition of A such that $g_i := f \restriction A_i$ is $\mathbf{\Sigma}^0_{n-m+1}$-measurable. As in the previous lemma, we can extend each g_i to a $\mathbf{\Sigma}^0_{n-m+1}$-measurable function $\tilde{g}_i : \tilde{A}_i \to Y$ with domain $\tilde{A}_i \in \mathbf{\Pi}^0_{n-m+2}(X) \subseteq \mathbf{\Pi}^0_{n-1}(X)$. Therefore, since

$$\mathrm{graph}(f) = \{(x, f(x)) \mid x \in A\} \in \mathbf{\Sigma}^0_m(A \times Y)$$

it follows that $B_i = (\tilde{g}_i, \mathrm{id})^{-1}[\mathrm{graph}(f)] \in \mathbf{\Sigma}^0_n(A)$ because $\tilde{g}_i$ $\mathbf{\Sigma}^0_{n-m+1}$-measurable with $\mathrm{dom}(\tilde{g}_i) = \tilde{A}_i \in \mathbf{\Pi}^0_{n-1}(A)$ and $\mathrm{graph}(f) \in \mathbf{\Sigma}^0_m$. In this case, we conclude further decomposing the B_is in $\mathbf{\Delta}^0_{n-1}$ pieces. □

3.2 Weak Solecky Dichotomy for recursive spaces

3.2.1 The Solecki Dichotomy in Descriptive Set Theory

Definition 3.18. Given X and Y topological spaces, $f : X \to Y$ is **σ-continuous** if there exists a countable partition $\{A_n\}_{n\in\omega}$ of X such that $g_n = f \restriction A_n$ is continuous for all $n \in \omega$.

Notice that, following the terminology introduced in the previous section, f σ-continuous can be denoted by $f \in \mathrm{dec}(\mathbf{\Sigma}^0_1)$.

Remark 3.19. As observed in [Deb14] and similarly to Lemma 3.16, if X is separable metrizable space, Y Polish spaces, and f is Borel-measurable, one can assume, without loss of generality, that the witnesses $\{A_n\}_{n\in\omega}$ are in **Bor**. Using the terminology introduced in [GKN21]:

$$f \text{ is } \sigma\text{-continuous} \Leftrightarrow f \in \operatorname{dec}(\mathbf{\Sigma}^0_1, \mathbf{Bor})$$

Indeed, each function g_n admits a continuous extension $\tilde{g}_n$ defined on $\tilde{A}_n \in \mathbf{\Pi}^0_2(X)$ (see [Kec95, Theorem 3.8]). Therefore, f is continuous on the sets $B_n = \{x \in \tilde{A}_n \mid \tilde{g}_n(x) = f(x)\} \supseteq A_n$ and they are Borel because $B_n = (f, \tilde{g}_n)^{-1}[\Delta_Y]$ (where Δ_Y is the diagonal set of Y that is $\mathbf{\Pi}^0_1$ because Y is Hausdorff). Finally, from $\{B_n\}_{n\in\omega}$ we can extract a partition witnessing $f \in \operatorname{dec}(\mathbf{\Sigma}^0_1, \mathbf{Bor})$.

Definition 3.20. Given X and Y topological spaces, a function $\varphi : X \to Y$ is a **topological embedding** of X into Y if it is a homeomorphism between X and its range $\operatorname{ran}(\varphi)$.

Definition 3.21. Given X_f, Y_f, X_g, Y_g topological spaces and two functions $f : X_f \to Y_f$ and $g : X_g \to Y_g$ we say that f **topologically embeds** into g if there exist two topological embeddings $\varphi : X_f \to X_g$ and $\psi : Y_f \to Y_g$ such that $\forall x \in X_f(\psi \circ f(x) = g \circ \varphi(x))$.

$$\begin{array}{ccc} X_g & \xrightarrow{g} & Y_g \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\psi} \\ X_f & \xrightarrow{f} & Y_f \end{array}$$

Definition 3.22. A separable metrizable space A is an **analytic space** if it is the image of a total continuous function from the Baire space ω^ω.

Usually, the Solecki Dichotomy is stated as:

Theorem 3.23 ([PS12, Theorem 1.1])**.** Given A analytic space, Y separable metrizable and $f : A \to Y$ Borel function, then either f is σ-continuous or the Pawlikowski function $P : (\omega+1)^\omega \to \omega^\omega$ topologically embeds into f.

Where the Pawlikowski function is the function $P : (\omega+1)^\omega \to \omega^\omega$ defined as:

$$P(x)(n) = \begin{cases} x(n)+1 & \text{if } x(n) < \omega \\ 0 & \text{if } x(n) = \omega \end{cases}$$

in which $(\omega+1)^\omega$ is endowed with the product of the order topology on $\omega+1$.

3.2.2 The weak Solecki Dichotomy with the Turing Jump

We consider two weaker form of reducibility between functions:

Definition 3.24. Given X_f, Y_f, X_g, Y_g topological spaces and two functions $f : X_f \to Y_f$ and $g : X_g \to Y_g$, we say that f is **continuously reducible** to g ($f \leq_s g$) if there exist a total continuous function $\varphi : X_f \to X_g$ and a partial continuous function $\psi : Y_g \rightharpoonup Y_f$ such that $\forall x \in X_f (f(x) = \psi(g(\varphi(x))))$.

$$\begin{array}{ccc} X_g & \xrightarrow{g} & Y_g \\ {\scriptstyle\varphi}\uparrow & & \downarrow{\scriptstyle\psi} \\ X_f & \xrightarrow{f} & Y_f \end{array}$$

By the previous definition we have that $\text{ran}(g \circ \varphi) \subseteq \text{dom}(\psi)$.

Definition 3.25. Given X_f, Y_f, X_g, Y_g topological spaces and two functions $f : X_f \to Y_f$ and $g : X_g \to Y_g$, we say that f is **weakly (continuous) reducible** to g ($f \leq_w g$) if there exist two partial continuous functions $\varphi : X_f \rightharpoonup X_g$ and $\psi : Y_g \times X_f \rightharpoonup Y_f$ such that $\forall x \in X_f (f(x) = \psi(g(\varphi(x)), x))$.

If f topologically embeds into g, then f is continuously reducible to g; moreover, continuous reducibility implies f is weakly reducible to g.

As in [Lut21] and [Lut23, similar to Definition 1.3], we introduce a different version of the Turing jump $J_\omega : 2^\omega \to \omega^\omega$ defined as:

$$(J_\omega(x))(n) = \begin{cases} 0 & \text{if } \varphi_n^x(n) \uparrow \\ k & \text{if } \varphi_n^x(n)[k] \downarrow \end{cases} \forall n \in \omega$$

We say that J_ω is a version of the Turing jump because $\forall x \in 2^\omega (J_\omega(x) \equiv_T x')$.

Fact 2. Consider the usual Turing jump $J : 2^\omega \to 2^\omega$ and the modified version $J_\omega : 2^\omega \to \omega^\omega$, then:

1. J and J_ω are $\mathbf{\Sigma}_2^0$-measurable
2. J and J_ω are injective
3. J and J_ω are not σ-continuous

Proof. 1. Recall that the usual Turing Jump is defined on any $x \in 2^\omega$ as $J(x) = x' = \{e \in \omega \mid \varphi_e^x(e) \downarrow\}$. In particular, we have:

$$(J(x))(n) = \begin{cases} 0 & \text{if } \varphi_n^x(n) \uparrow \\ 1 & \text{if } \varphi_n^x(n) \downarrow \end{cases} \forall n \in \omega$$

To prove that J is $\mathbf{\Sigma}^0_2$-measurable, we consider the following subbase of the Cantor space:

$$\mathcal{B} = \{B_{n,i} \mid n \in \omega, i \in 2\} \text{ where } B_{n,i} = \{x \in 2^\omega \mid x(n) = i\}$$

We have

$$J^{-1}[B_{n,1}] = \{x \in 2^\omega \mid \varphi^x_n(n) \downarrow\}$$

To estimate the complexity of this set we use the Finite use property (Proposition 2.37), thus:

$$J^{-1}[B_{n,1}] = \bigcup\{N_{s_x} \mid s_x \in \omega^{<\omega} \text{ is the minimal witness for } \varphi^x_n(n) \downarrow\} \in \mathbf{\Sigma}^0_1$$

Similarly, we have

$$J^{-1}[B_{n,0}] = \{x \in 2^\omega \mid \varphi^x_n(n) \uparrow\} = 2^\omega \setminus J^{-1}[B_{n,1}] \in \mathbf{\Pi}^0_1$$

Therefore, the preimage of each element of the basis $\{N_s \mid s \in 2^{<\omega}\}$ is in $\mathbf{\Delta}^0_2(2^\omega)$ and hence J is $\mathbf{\Sigma}^0_2$-measurable.
Similarly, one can prove that $J_\omega : 2^\omega \to \omega^\omega$ is $\mathbf{\Sigma}^0_2$-measurable.

2. To prove that $J_\omega : 2^\omega \to \omega^\omega$ is injective, we consider for each oracle $x \in 2^\omega$ the Turing Machine M^x_j that on any input:

   - read the first j cells of the tape containing x
   - then go right for other $x(j) \in 2$ steps and then halts.

   In particular, on any input a, the machine M^x_j halts after $j + x(j)$ steps, that is $M^x_j(a)[j + x(j)] \downarrow$.
   Let $(n^x_j)_{j\in\omega}$ be the enumeration of the codes of the machines $(M^x_j)_{j\in\omega}$.
   As we consider a standard coding (of all the Turing Machines) that is independent by the oracle, we get $\forall x, z \in 2^\omega \forall j \in \omega (n^x_j = n^z_j)$, so we call them just $(n_j)_{j\in\omega}$.
   Now given $x, z \in 2^\omega$ such that $J_\omega(x) = J_\omega(z)$, we have:

$$\forall j \in \omega \exists s \in \omega (\varphi^x_{n_j}(n_j)[s] \downarrow \Leftrightarrow \varphi^z_{n_j}(n_j)[s] \downarrow) \Rightarrow$$
$$\forall j \in \omega \exists s \in \omega (j + x(j) = s = j + z(j)) \Rightarrow$$
$$\forall j \in \omega (x(j) = z(j)) \Rightarrow x = z$$

   We observe that a similar argument can be used to prove that $J : 2^\omega \to 2^\omega$ is injective, indeed it suffices to consider (for any oracle $x \in 2^\omega$) the Turing Machine $M^x_{\langle n,k\rangle}$ that on any input:

   - read the value $x(n)$ on the tape containing x
   - then halts if $x(n) = k$ and loops otherwise.

3. We prove that

   Claim 11. A function $f : 2^\omega \to 2^\omega$ is σ-continuous if and only if $\exists w \in 2^\omega \forall x \in 2^\omega (f(x) \leq_T x \oplus w)$.[3]

   Proof of the Claim. $\Rightarrow$ As f is σ-continuous, there is a partition $\{A_n\}_{n\in\omega}$ of 2^ω such that $f_n := f \upharpoonright A_n$ is continuous. Therefore, for each $n \in \omega$ there exists an oracle $w_n \in \omega^\omega$ and a partial computable $\tilde{f}_n : 2^\omega \rightharpoonup 2^\omega$ such that $\forall x \in A_n(f_n(x) = \tilde{f}_n(x, w_n))$. That is $\forall x \in A_n(f(x) \leq_T x \oplus w_n)$. Finally, considering $w = \bigoplus_{n\in\omega} w_n$ we have that $\forall x \in 2^\omega(f(x) \leq_T x \oplus w)$.

   $\Leftarrow$ We define $A_n = \{x \in 2^\omega \mid f(x) = \Phi_n^{(2^\omega \times 2^\omega), 2^\omega}(x, w)\}$. The family $\{A_n\}_{n\in\omega}$ is a covering of 2^ω and $f \upharpoonright A_n$ is computable relatively to w, hence f is σ-continuous. □

   As the equivalent condition to σ-continuity does not holds for both versions of the Turing Jump, the thesis follows. □

Proposition 3.26 ([Lut21, Proposition 2.28])**.** We have that:

1. The modified Turing jump $J_\omega : 2^\omega \to \omega^\omega$ topologically embeds into the Pawlikowski function P.

2. The usual Turing jump $J : 2^\omega \to 2^\omega$ is continuously reducible to $J_\omega : 2^\omega \to \omega^\omega$.

Proof. 1. We define the function $\varphi : 2^\omega \to (\omega + 1)^\omega$ as

$$\varphi(x)(n) = \begin{cases} \omega & \text{if } \varphi_n^x(n) \uparrow \\ k & \text{if } \varphi_n^x(n)[k] \downarrow \end{cases}$$

and $\psi : \omega^\omega \to \omega^\omega$ as

$$\psi(x)(n) = \begin{cases} 0 & \text{if } x(n) = 0 \\ x(n) + 1 & \text{otherwise} \end{cases}$$

By construction $\forall x \in 2^\omega(\psi \circ J_\omega(x) = P \circ \varphi(x))$. Moreover, ψ is a topological embedding, and hence we only have to check that φ is a topological embedding. We observe that

φ **injective:** By the previous fact J_ω injective. Hence so is $P \circ \varphi = \psi \circ J_\omega$, and thus φ is injective.

[3] Clearly, the same statement holds if one (or both) between domain and codomain of f is (or are) ω^ω. We skip the details as we show in Proposition 3.31 how to extend the same results to functions between recursive spaces.

$\varphi : 2^\omega \to (\omega+1)^\omega$ **continuous:** Given $(x_i)_{i\in\omega}$ sequence in 2^ω which converges to a fixed $x \in 2^\omega$, we prove that $\varphi(x_i) \underset{i\to\infty}{\longrightarrow} \varphi(x)$. Observe that for all $n \in \omega$ there are two possibilities:

- If $\varphi_n^x(n)[k]\downarrow$ then, by the finite use property (Proposition 2.37) there is some $s \in 2^{<\omega}$ such that $s \prec x$ witnessing this. Since $\exists m \in \omega \forall j \geq m (s \prec x_j)$, then the n-th coordinates of $\varphi(x_j)$ will eventually be k.
- If $\varphi_n^x(n)\uparrow$ then, again by the finite use property, for each $k \in \omega$, there is some $s \prec x$ witnessing that $\varphi_n^x(n)$ takes more than k steps to converge and so $\exists m \in \omega \forall j \geq m (\varphi(x_j)(n) \geq k)$ (possibly ω).

Thus, for each coordinate, w.r.t. the order topology of $(\omega+1)$, the sequence $(\varphi(x_i)(n))_{i\in\omega}$ converges to $\varphi(x)(n)$ and hence $(\varphi(x_i))_{i\in\omega}$ converges to $\varphi(x)$.

As φ is a continuous injection with compact domain, it is necessarily a topological embedding.

2. We consider the continuous functions $\varphi = \mathrm{id} : 2^\omega \to 2^\omega$ and $\psi : \omega^\omega \to 2^\omega$ as

$$\psi(x)(n) = \begin{cases} 0 & \text{if } x(n) = 0 \\ 1 & \text{otherwise} \end{cases}$$

Then $\forall x \in 2^\omega (J(x) = \psi \circ J_\omega \circ \varphi(x))$. □

Moreover, since both versions of the Turing Jump are not σ-continuous, as corollary of the Solecki Dichotomy we obtain:

Theorem 3.27. Given A analytic space, Y separable metrizable and $f : A \to Y$ Borel function, then either f is σ-continuous or the Turing jump $J_\omega : 2^\omega \to \omega^\omega$ topologically embeds into f.

In particular, we get the following weak version of the Solecki Dichotomy:

Theorem 3.28. Given A analytic space, Y separable metrizable and $f : A \to Y$ Borel function, then either f is σ-continuous or $J \leq_W f$.

We now use the Posner-Robinson Theorem 2.60 for continuous degrees to get an alternative proof of the previous statement in the case of recursively presented Polish spaces:

Theorem 3.29. Given $\mathcal{X}$ recursively presented Polish space, $\mathcal{Y}$ recursive space and $f : X \to Y$ Borel function, then either f is σ-continuous or $J \leq_W f$.

To obtain this result, we modify an argument of Patrick Lutz in [Lut23] for Borel measurable function from and into the Baire space. His proof uses a game argument to apply Borel Determinacy and obtain the dichotomy. We need to modify the game, using admissible representations for recursive spaces. However, we follow quite closely the original proof scheme.

3.2.3 The generalization of Lutz's proof

Consider two functions $f : \omega^\omega \to \omega^\omega$[4] and $g : X_g \to Y_g$ (where $\mathcal{X}_g$ and $\mathcal{Y}_g$ are recursive spaces) we can generalize the game presented in [Lut23, Section 2] in a game called $G_M(f, g)$, played as follows:

Player 1		x_0	x_1	...
Player 2	e	$V_{b_0}^{\mathcal{X}_g}$, z_0	$V_{b_1}^{\mathcal{X}_g}$, z_1	...

where $(V_j^{\mathcal{X}_g})_{j\in\omega}$ is the enumeration of the basis of $\mathcal{X}_g$ as a recursive space. Player 2 first plays a code $e \in \omega$ corresponding to Σ_1^0-recursive function on its domain from $\mathcal{Y}_g \times \omega^\omega \times \omega^\omega$ to ω^ω. For the rest of the game, Player 1 plays a real $x \in \omega^\omega$ and Player 2 plays two reals, $b \in \omega^\omega$ and $z \in \omega^\omega$ such that $b \in \mathrm{dom}(\rho_{\mathcal{X}_g})$. In particular, similarly to the original game, on every turn Player 1 plays one more digit of the real $x = x_0 x_1 \dots$ and Player 2 plays one more digit of the reals $b = b_0 b_1 \dots$ and $z = z_0 z_1 \dots$.[5] Player 2 wins if and only if b is an actual name for an element $y \in \mathcal{X}_g$ (i.e. $y = \rho_{\mathcal{X}_g}(b) = \bigcap_{n\in\omega} V_{b_n}^{\mathcal{X}_g}$) and $f(x) = \Phi_e^{(\mathcal{Y}_g \times \omega^\omega \times \omega^\omega), \omega^\omega}(g(\rho_{\mathcal{X}_g}(b)), x, z)$.
If f and g are Borel functions and the domain of the representation $\rho_{\mathcal{X}_g}$ is Borel, the payoff of the game $G_M(f, g)$ is Borel, since Borel is its complement:

$$A = \{(x, e, b, z) \mid f(x) = \Phi_e^{(Y_g \times \omega^\omega \times \omega^\omega), \omega^\omega}(g(\rho_{\mathcal{X}_g}(b)), x, z) \land b \in \mathrm{dom}(\rho_{\mathcal{X}_g})\}$$

The following lemmas are extensions of results presented in [Lut23, Section 2] to the context of recursive spaces.

Lemma 3.30 ([Lut23, Lemma 2.1])**.** If Player 2 has a winning strategy in $G_M(f, g)$, then $f \leq_w g$.

[4] If the domain of f is the Cantor space the definition of the game does not really change.

[5] Here there is a slight difference from the original game: Player 2 cannot do a turn in which they delays the play of a digit of b. Because we do not need this feature for our result.

Proof. Suppose that Player 2 wins $G_M(f, g)$ using the strategy τ.
We describe how to define the two partial continuous functions $\varphi : \omega^\omega \rightharpoonup X_g$ and $\psi : Y_g \times \omega^\omega \rightharpoonup \omega^\omega$.

φ: Given $x \in \omega^\omega$, we play the game $G_M(f, g)$ using the digits of x as Player 1's moves and τ to generate b played by Player 2. Hence, we define $\tilde{\varphi}(x) = \bigcap_{i \in \omega} N_{b \upharpoonright i}$. The function $\tilde{\varphi}$ is continuous[6] because the family defined as

$$\mathcal{S} = \{N_{b \upharpoonright m} \mid m \in \omega \wedge \exists x \in \omega^\omega \exists e \in \omega \exists z \in \omega^\omega ((e, b, z) = \tau * (x))\}$$

is a Lusin scheme, where $\tau * (x)$ is the play of Player 2 according to the strategy τ in response to Player 1 playing x.
Indeed we can relabel the element of $\mathcal{S}$ in the following way:

- $B_\varnothing = N_\varepsilon = X$ (where $\varepsilon \in \omega^{<\omega}$ is the empty string).
- $B_s = N_{b^s}$ where $b^s = (b_0^s, \dots, b_{\ell(s)-1}^s)$ and b_i^s is the b_i played by the Player 2 after $i \leq \ell(s)$ turns following the strategy τ against the Player 1 that played $s \in \omega^{<\omega}$.

In this way, $\forall s \in \omega^{<\omega} \forall n \in \omega (B_{s^\frown n} \subseteq B_s)$ and $\forall x \in \omega^\omega (\text{diam}(B_{x \upharpoonright n}) \underset{n \to \infty}{\longrightarrow} 0)$ Moreover:

$$D_{\mathcal{S}} = \left\{ x \in \omega^\omega \,\middle|\, \bigcap_{n \in \omega} B_{x \upharpoonright n} \neq \varnothing \right\} = \omega^\omega$$

and $\tilde{\varphi}$ is the function induced by it, thus it is continuous. Therefore we define φ as $\varphi(x) = \rho_{\mathcal{X}_g} \circ \tilde{\varphi}(x)$. Notice that, since τ is a winning strategy, $\tilde{\varphi}(x)$ is a name for some element in $\mathcal{X}_g$.

ψ: Given $w \in Y_g$ and $x \in \omega^\omega$, we play the game $G_M(f, g)$ using the digits of x as Player 1's moves and τ to generate e and z played by Player 2. Then we define ψ as $\psi(w, x) = \Phi_e^{(\mathcal{Y}_g \times \omega^\omega \times \omega^\omega), \omega^\omega}(w, x, z)$.

Since τ is a winning strategy for Player 2, we get:

$$\forall x \in \omega^\omega (f(x) = \psi(g(\varphi(x)), x)) \qquad \square$$

Proposition 3.31 ([Lut23, Observation 2.2] and [GKN21, observed after Remark 3.5])**.** Given $\mathcal{X}_f$ and $\mathcal{Y}_f$ recursive spaces:

$$f : X_f \to Y_f \ \sigma\text{-continuous} \Leftrightarrow \exists w \in 2^\omega \forall x \in X_f (f(x) \leq_M (x \oplus w))$$

[6] Actually, one can also prove that $\tilde{\varphi}$ is Lipschitz.

Proof.

- ⇒ f σ-continuous means that there is a partition $\{A_n\}_{n\in\omega}$ of X_f such that $f_n := f \upharpoonright A_n$ is continuous. By Theorem 2.8, for each $n \in \omega$ there is an $F_n : \omega^\omega \rightharpoonup \omega^\omega$ continuous such that

$$f_n \circ \rho_{\mathcal{X}_f}(\alpha) = \rho_{\mathcal{Y}_f} \circ F_n(\alpha) \quad \forall \alpha \in \operatorname{dom}(\rho_{\mathcal{X}_f})$$

  Moreover, F_n partial continuous implies that there exists an oracle $w_n \in \omega^\omega$ and a computable $\tilde{F}_n$ such that $\forall \alpha \in \omega^\omega(F_n(\alpha) = \tilde{F}_n(\alpha, w_n))$. We prove that $\forall x \in A_n(f(x) \leq_M x \oplus w_n)$. In particular, the computable function that witnesses this reduction is the function $g_n : \mathcal{X}_f \times \omega^\omega \rightharpoonup \mathcal{Y}_f$ induced by $\tilde{F}_n(*, \rho_{\omega^\omega}(*)) : \omega^\omega \times \omega^\omega \rightharpoonup \omega^\omega$, that is the function that satisfies $\forall \alpha \in \operatorname{dom}(\rho_{\mathcal{X}_f}) \forall \beta \in \operatorname{dom}(\rho_{\omega^\omega})$

$$g_n \circ \rho_{\mathcal{X}_g \times \omega^\omega}(\alpha \oplus \beta) = g_n(\rho_{\mathcal{X}_f}(\alpha), \rho_{\omega^\omega}(\beta)) = \rho_{\mathcal{Y}_f} \circ \tilde{F}_n(\alpha, \rho_{\omega^\omega}(\beta))$$

  Indeed: $\forall x \in A_n(g_n(x, w_n) = f_n(x))$.
  Therefore, considering $w = \bigoplus_{n\in\omega} w_n$ we have $\forall x \in X_f(f(x) \leq_M x \oplus w)$.

- ⇐ We define $A_n = \{x \in X_f \mid f(x) = \Phi_n^{(\mathcal{X}_f \times \omega^\omega), \mathcal{Y}_f}(x, w)\}$. The family $\{A_n\}_{n\in\omega}$ is a covering of X_f and $f \upharpoonright A_n$ is computable relatively to the oracle w, hence f is σ-continuous. □

Moreover, we are able to prove an analogue of [Lut23, Lemma 2.3]:

Lemma 3.32 ([Lut23, Lemma 2.3])**.** If Player 1 has a winning strategy in $G_M(J_\omega, g)$, then g is σ-continuous.[7]

Proof. Suppose that Player 1 wins $G_M(J_\omega, g)$ using the strategy τ.
Consider the function $\tilde{g} := g \circ \rho_{\mathcal{X}_g} : \omega^\omega \rightharpoonup Y_g$. Recall that a set $A \subseteq \omega^\omega$ is $\rho_{\mathcal{X}_g}$*-saturated* if $A = \rho_{\mathcal{X}_g}^{-1}[\rho_{\mathcal{X}_g}[A]]$.

Claim 12. If $\tilde{g}$ is countable union of continuous functions defined on $\rho_{\mathcal{X}_g}$-saturated domains, then g is σ-continuous.

Proof of the Claim. Suppose that $\{B_n\}_{n\in\omega}$ is a covering of $\operatorname{dom}(\tilde{g}) = \operatorname{dom}(\rho_{\mathcal{X}})$ by $\rho_{\mathcal{X}_g}$-saturated sets such that $\tilde{g}_n = \tilde{g} \upharpoonright B_n$ is continuous. We prove that $\{\rho_{\mathcal{X}_g}[B_n]\}_{n\in\omega}$ is a covering of X_g such that $g \upharpoonright \rho_{\mathcal{X}_g}[B_n]$ is continuous.[8]
Indeed, given $U \subseteq Y_g$ open, there is an open set $V \subseteq \omega^\omega$ such that:

$$\tilde{g}_n^{-1}[U] = (g \circ (\rho_{\mathcal{X}_g} \upharpoonright B_n))^{-1}[U] = (\rho_{\mathcal{X}_g} \upharpoonright B_n)^{-1} \circ g^{-1}[U] = V \cap B_n$$

[7]This theorem as the following results can also be stated with the usual Turing Jump $J : 2^\omega \to 2^\omega$, in fact the proof is exactly the same.

[8]From here we are able to extract a partition witnessing that g is σ-continuous, and if $\mathcal{Y}_g$ is Polish recursive we can also find a partition with Borel domains as seen in Remark 3.19.

since $\tilde{g} \upharpoonright B_n$ is continuous, therefore

$$(g \upharpoonright \rho_{\mathcal{X}_g}[B_n])^{-1}[U] = \rho_{\mathcal{X}_g}[V \cap B_n] = \rho_{\mathcal{X}_g}[V] \cap \rho_{\mathcal{X}_g}[B_n]$$

where the last equality holds because B_n is $\rho_{\mathcal{X}_g}$-saturated. Since $\rho_{\mathcal{X}_g}$ is open and surjective, then $\rho_{\mathcal{X}_g}[V]$ is open and hence $g \upharpoonright \rho_{\mathcal{X}_g}[B_n]$ is continuous. □

Claim 13. If $\forall b \in \mathrm{dom}(\tilde{g})(\tilde{g}(b) \leq_M \rho_{\mathcal{X}_g}(b) \oplus \tau)$, then $\tilde{g}$ is countable union of continuous functions defined on $\rho_{\mathcal{X}_g}$-saturated domains.

Proof of the Claim. We set:

$$\begin{aligned} B_n &= \{b \in \mathrm{dom}(\rho_{\mathcal{X}_g}) \mid \tilde{g}(b) = \Phi_n^{(\mathcal{X}_g \times \omega^\omega), \mathcal{Y}_g}(\rho_{\mathcal{X}_g}(b), \tau)\} \\ &= \{b \in \mathrm{dom}(\rho_{\mathcal{X}_g}) \mid g \circ \rho_{\mathcal{X}_g}(b) = \Phi_n^{(\mathcal{X}_g \times \omega^\omega), \mathcal{Y}_g}(\rho_{\mathcal{X}_g}(b), \tau)\} \end{aligned}$$

$\{B_n\}_{n \in \omega}$ is a countable covering of $\mathrm{dom}(\rho_{\mathcal{X}_g})$ and $\tilde{g} \upharpoonright B_n$ is continuous. Moreover, each B_n is $\rho_{\mathcal{X}_g}$-saturated as

$$\forall d \in \omega^\omega \forall b \in B_n(\rho_{\mathcal{X}_g}(d) = \rho_{\mathcal{X}_g}(b) \Rightarrow d \in B_n)$$ □

We prove that, under the hypothesis of the lemma, $\forall b \in \mathrm{dom}(\tilde{g})(\tilde{g}(b) \leq_M \rho_{\mathcal{X}_g}(b) \oplus \tau)$ and hence g is σ-continuous. To prove this, we show that if not, then τ is not actually a winning strategy for Player 1.
So, towards a contradiction, suppose that there is some $b \in \mathrm{dom}(\tilde{g})$ such that $\tilde{g}(b) \not\leq_M \rho_{\mathcal{X}_g}(b) \oplus \tau$. Therefore by Corollary 2.34 $\exists w \in 2^\omega$ such that $\tilde{g}(b) \not\leq_M w$ and $\rho_{\mathcal{X}_g}(b) \oplus \tau \leq_M w$. Using Posner-Robinson Theorem 2.60 relative to w, we can find some $v \in 2^\omega$ such that $\tilde{g}(b) \oplus w \oplus v \geq_M (v \oplus w)'$.
Another difference with the original proof is that now, if we let play to the Player 2 b and $z = w \oplus v$, then this z cannot compute the moves of the (hypothetically) winning play of the Player 1 (because it cannot compute b). To overcome this we need to choose another name to play, in particular the one given by point 4 of Fact 1. Indeed, we have that:

$$\rho_{\mathcal{X}_g}(b) \leq_M w \Leftrightarrow \exists p \in \omega^\omega(\rho_{\mathcal{X}_g}(p) = \rho_{\mathcal{X}_g}(b) \wedge p \leq_T w)$$

Notice that $\rho_{\mathcal{X}_g}(p) = \rho_{\mathcal{X}_g}(b)$, implies also: $\tilde{g}(p) \not\leq_M \rho_{\mathcal{X}_g}(p) \oplus \tau$, $\tilde{g}(p) \not\leq_M w$, $\rho_{\mathcal{X}_g}(p) \oplus \tau \leq_M w$, and $\tilde{g}(p) \oplus w \oplus v \geq_M (v \oplus w)'$. We now are ready to explain how to win while playing as Player 2 in $G_M(J_\omega, g)$ against the strategy τ. We play as follows:

- First we play some number $e \in \omega$, which we explain how to choose later.
- Then we ignore Player 1's moves and play the reals p and $z = v \oplus w$.

Note that from z we can compute Player 1's moves (since the x played by Player 1 is in Cantor) because z computes both Player 1's strategy τ and all of Player 2's moves. That is: if $x \in 2^\omega$ is played by Player 1 according to τ, then $x \leq_M z$.
Hence by our choice of z we have:

$$\tilde{g}(p) \oplus z \equiv_M \tilde{g}(p) \oplus w \oplus v \geq_M (w \oplus v)' \geq_M x' \equiv_M J_\omega(x)$$

From this we can conclude in the same way as in the original proof. In particular, as in the original proof, the computation of $J_\omega(x)$ from $\tilde{g}(p) \oplus z$ depends also on the value $e \in \omega$ played by Player 2, but in an uniform way. That is, said $\tau * (e, p, z) \in 2^\omega$ the element played by the strategy τ in response to the play of Player 2 corresponding to (e, p, z), there is an $a \in \omega$ such that

$$\forall e \in \omega \left(\Phi_a^{\mathcal{Y}_g \times \omega^\omega \times \omega, \omega^\omega}(\tilde{g}(p), z, e) = J_\omega(\tau * (e, p, z)) \right)$$

Therefore using Kleene's Recursion Theorem 1.20 for recursive spaces, there is an $\tilde{e} \in \omega$ such that:

$$\Phi_{\tilde{e}}^{\mathcal{Y}_g \times \omega^\omega, \omega^\omega}(g \circ \rho_{\mathcal{X}_g}(p), z) = \Phi_a^{\mathcal{Y}_g \times \omega^\omega \times \omega, \omega^\omega}(\tilde{g}(p), z, \tilde{e}) = J_\omega(\tau * (\tilde{e}, p, z))$$

Therefore, if Player 2 starts playing with $\tilde{e}$ they can win against τ, and hence τ is not a winning strategy. ↯
Thus, using the claims at the beginning of the proof, it follows that g is σ-continuous. □

Finally using the Borel determinacy, we can prove the following weak version of the Solecki Dichotomy:

Theorem 3.33. Given $\mathcal{X}$, $\mathcal{Y}$ recursive spaces such that $\rho_{\mathcal{X}}$ has Borel domain and $g : \mathcal{X} \to \mathcal{Y}$ Borel function, then either g is σ-continuous or $J_\omega \leq_w g$.

Proof. We play the game $G_M(J_\omega, g)$. By Borel determinacy, either Player 1 or Player 2 has a winning strategy. Therefore, we have two cases:

Player 1 wins then, by Lemma 3.32, g is σ-continuous.

Player 2 wins then, by Lemma 3.30, $J_\omega \leq_w g$. □

Clearly, under the Axiom of Determinacy AD, this result generalizes to all recursive spaces and functions.

What about recursively presented Polish spaces?

The careful reader, will notice that we initially stated the theorem for recursively presented Polish spaces and, moreover, we never mentioned the use of the Axiom of Determinacy. So where is the trick? The fact that gives us the statement for recursively presented Polish spaces is that, given such a space $\mathcal{X}$, the domain of the admissible representation $\rho_{\mathcal{X}} : \omega^\omega \rightharpoonup \mathcal{X}$ is Borel. Indeed, it is $\mathbf{\Sigma}^1_1$ since:

$$\mathrm{dom}(\rho_{\mathcal{X}}) = \{b \in \omega^\omega \mid \exists y \in X(\forall n \in \omega(y \in V^{\mathcal{X}}_n \Leftrightarrow \exists j \in \omega(n = b(j))))\}$$

Moreover we can consider also the $\mathbf{\Pi}^1_1$ set:

$$\begin{aligned}\tilde{\mathrm{dom}}(\rho_{\mathcal{X}}) = \{b \in \omega^\omega \mid &\forall y \in X[\forall i \in \omega(y \in V^{\mathcal{X}}_{b(i)}) \Rightarrow \forall n \in \omega(y \in V^{\mathcal{X}}_n \Leftrightarrow \exists j \in \omega(n = b(j)))]\wedge\\ &\forall n \in \omega \exists m \in \omega(S(b(m), b(n)))\wedge\\ &\forall n \in \omega \forall m \in \omega \exists p \in \omega(R(b(m), b(n), b(p)))\wedge\\ &\forall k \in \omega \forall n \in \omega \exists m \in \omega(m \geq n \wedge q((m)_1) < 2^{-k})\}\end{aligned}$$

Where $(q_n)_{n\in\omega}$ is the fixed effective enumeration of $\mathbb{Q}^+$, R is the semirecursive predicate that makes $\mathcal{X}$ basic and S is the semirecursive predicate that makes $\mathcal{X}$ recursively regular (together with T). Thus, as ω^ω is Polish, it suffices to prove that $\mathrm{dom}(\rho_{\mathcal{X}}) = \tilde{\mathrm{dom}}(\rho_{\mathcal{X}})$ and hence $\mathrm{dom}(\rho_{\mathcal{X}}) \in \mathbf{\Delta}^1_1(\omega^\omega) = \mathbf{Bor}(\omega^\omega)$. Towards this direction, we first give some remarks on the added conditions. The first is that they are not restrictive if $b \in \omega^\omega$ is a name for $y \in X$.

Lemma 3.34. Given $\mathcal{X}$ recursive space and $b \in \omega^\omega$ $\rho_{\mathcal{X}}$-name of $y \in \mathcal{X}$ (that is $\forall n \in \omega(y \in V^{\mathcal{X}}_n \Leftrightarrow n \in \mathrm{ran}(b))$), then the followings hold:

A. $\forall n \in \omega \exists m \in \omega(S(b(m), b(n)))$

B. $\forall n \in \omega \forall m \in \omega \exists p \in \omega(R(b(m), b(n), b(p)))$

Moreover, if $\mathcal{X}$ is also a recursively presented metric space holds:

$$\forall k \in \omega \forall n \in \omega \exists m \in \omega(m \geq n \wedge q((m)_1) < 2^{-k})$$

Proof. A. For every $n \in \omega$ then $y \in V^{\mathcal{X}}_{b(n)}$, thus by the first condition of recursively regular space:

$$\exists i(y \in V^{\mathcal{X}}_i \wedge S(i, b(n)))$$

and $\exists m \in \omega(i = b(m))$ because b is a name of y.

B. For every $n \in \omega$ we have $y \in V^{\mathcal{X}}_{b(n)}$, thus $y \in V^{\mathcal{X}}_{b(n)} \cap V^{\mathcal{X}}_{b(m)}$ and by definition of basic space:

$$\exists i(y \in V^{\mathcal{X}}_i \wedge R(b(m), b(n), i))$$

hence we conclude as in the previous point.

Finally, the last part follows because a $\rho_{\mathcal{X}}$-name enumerate all the balls centered in any points of the dense set enumerated by $\boldsymbol{r}$ and with radius any positive rationals. □

Moreover, these conditions guarantee the non-emptiness of the condition $\forall i \in \omega (y \in V^{\mathcal{X}}_{b(i)})$ as showed by the following lemmas:

Lemma 3.35. Given $\mathcal{X}$ recursive space and $b \in \omega^\omega$ such that $\forall n \in \omega \exists m \in \omega (S(b(m), b(n)))$ then

$$\bigcap_{i \in \omega} \hat{V}^{\mathcal{X}}_{b(i)} = \bigcap_{i \in \omega} V^{\mathcal{X}}_{b(i)}$$

where $\hat{V}^{\mathcal{X}}_j$ is the closure of the $V^{\mathcal{X}}_j$ in the effective basis.

Proof. One inclusion is clear, for the other consider $y \in \bigcap_{i \in \omega} \hat{V}^{\mathcal{X}}_{b(i)}$. By our assumptions $\forall n \in \omega \exists m \in \omega (S(b(m), b(n)))$, thus by the second condition in the definition of recursively regular space we have that fixed any $n \in \omega$ there is an $m \in \omega$ such that $y \in \hat{V}^{\mathcal{X}}_{b(m)} \subseteq V^{\mathcal{X}}_{b(n)}$. Hence $\forall i \in \omega (y \in V^{\mathcal{X}}_{b(i)})$. □

Lemma 3.36. Given $\mathcal{X}$ recursively presented Polish space and $b \in \omega^\omega$ such that

- $\forall n \in \omega \forall m \in \omega \exists p \in \omega (R(b(m), b(n), b(p)))$
- $\forall k \in \omega \forall n \in \omega \exists m \in \omega (m \geq n \wedge q((m)_1) < 2^{-k})$

then $\bigcap_{i \in \omega} \hat{V}^{\mathcal{X}}_{b(i)} \neq \varnothing$.

Proof. By our construction, since we consider balls with positive radius, we have that each element of the effective basis $V^{\mathcal{X}}_{b(n)}$ is non-empty.
To prove the non-emptiness of the intersection we will use the Cantor Intersection Lemma hence, since such family contains sets of arbitrarily small diameter, we only need to prove that: $\forall n \in \omega \left(\bigcap_{i \leq n} \hat{V}^{\mathcal{X}}_{b(i)} \neq \varnothing \right)$.
We prove by induction something more: for every $n \in \omega$ and subset of $B \subseteq \operatorname{ran}(b)$ of cardinality $n+1$ the intersection ranging over this set of balls in the effective basis is non-empty. The base case is obvious, thus we suppose that the result holds for all subset of cardinality n and prove it for all B of cardinality $n+1$.
We consider $b(i), b(j) \in B$, by our assumptions:

$$\exists p \in \omega (R(b(i), b(j), b(p)))$$

thus $V^{\mathcal{X}}_{b(p)} \subseteq V^{\mathcal{X}}_{b(i)} \cap V^{\mathcal{X}}_{b(j)}$ and $V^{\mathcal{X}}_{b(p)} \neq \varnothing$. Hence we consider the set $B' = (B \setminus \{b(i), b(j)\}) \cup \{b(p)\}$ of cardinality n and we apply the inductive hypothesis on it. Thus the thesis follows because $\varnothing \neq \bigcap_{b(i) \in B'} V^{\mathcal{X}}_{b(i)} \subseteq \bigcap_{b(i) \in B} V^{\mathcal{X}}_{b(i)}$. □

Remark 3.37. We point that we used the conditions with S and R (that make any recursively presented Polish space a recursively regular), instead of giving directly conditions with the metric because we originally wanted to prove such result for recursive Polish spaces. However, to apply the Cantor intersection Lemma, we are forced to use the diameter of the intersection (that has to decrease to zero) and hence we have to consider metric spaces and not of metrizable spaces.

Now, we are ready to prove that $\mathrm{dom}(\rho_{\mathcal{X}}) = \tilde{\mathrm{dom}}(\rho_{\mathcal{X}})$.

- $\subseteq$: Given $b \in \mathrm{dom}(\rho_{\mathcal{X}})$, then exists $y \in \bigcap_{i\in\omega} V^{\mathcal{X}}_{b(i)}$ and since b is a name and $\mathcal{X}$ is T_0, this y is unique. Hence y is the only value that satisfies the first condition of $\tilde{\mathrm{dom}}(\rho_{\mathcal{X}})$ w.r.t. b. Moreover, as b is a name it satisfies the other conditions (as shown in Lemma 3.34).
- $\supseteq$: Given $b \in \tilde{\mathrm{dom}}(\rho_{\mathcal{X}})$ by the last two lemmas we have that $\bigcap_{i\in\omega} V^{\mathcal{X}}_{b(i)} \neq \varnothing$. Therefore, the first condition is not vacuously true, then considered the y verifying the hypothesis of the first condition we have that b is a name for it. Thus $b \in \mathrm{dom}(\rho_{\mathcal{X}})$.

This allows us to state the following weak version of the Solecki Dichotomy:

Theorem 3.38. Given $\mathcal{X}$ recursively presented Polish space, $\mathcal{Y}$ recursive space and $g : X \to Y$ Borel function, then either g is σ-continuous or $J_\omega \leq_W g$.

Remark 3.39 (And for Polish spaces in general?)**.** Similarly to how we did in the first section of this chapter, considering an oracle $\varepsilon \in \omega^\omega$ strong enough to make a Polish space become ε-recursively presented Polish space we can repeat the same proof and obtain the weak form of the Solecki Dichotomy for every Borel function between Polish spaces.

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