Around the decomposability of Borel functions

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The original problem

Question (Luzin (early 1900s))

Is every Borel function between Polish spaces decomposable as countable union of continuous functions?

This question has been answered negatively by:

1. Novikov
2. Keldysh (1934) with a generalized level-by-level answer, i.e. she found for each $1 \leqslant \alpha < \omega_1$ a function $\mathbf{\Sigma}^0_{\alpha+1}$-measurable which cannot be decomposed as countable union of functions of lower complexity.

Notation

We write $f \in \mathrm{dec}(\mathbf{\Sigma}^0_\alpha)$ if f is decomposable as countable union of $\mathbf{\Sigma}^0_\alpha$-measurable functions (on their domains).

The Solecki Dichotomy

Although such decomposability of Borel functions is false in general, the non decomposable functions can be characterized using

Theorem (Solecki Dichotomy)

Given X analytic space, Y separable metrizable and $f : X \to Y$ Borel; then either $f \in \mathrm{dec}(\boldsymbol{\Sigma}^0_1)$ or f "contains" the Pawlikowski *function $P : (\omega+1)^\omega \to \omega^\omega$, defined as:*

$$P(\eta)(n) = \begin{cases} 0 & \text{if } \eta(n) = \omega \\ \eta(n)+1 & \text{otherwise} \end{cases}$$

This result was first proved by Solecki (1998) for Baire class 1 functions and then extended to all Borel function by Zapletal (2004) and by Pawlikowski and Sabok to all functions with analytic graphs (2012).

Fact (Carroy, Lutz)

The Pawlikowski function is "equivalent" to the **Turing Jump** *i.e. the function*

$$\begin{aligned} J : \omega^\omega &\to 2^\omega \\ x &\mapsto J(x) = x' = \{e \in \omega \mid \varphi^x_e(e)\downarrow\} \end{aligned}$$

The Solecki Dichotomy (more precisely)

Definition

Given X_f, Y_f, X_g, Y_g topological spaces and functions $f : X_f \to Y_f$ and $g : X_g \to Y_g$, we say that f **embeds topologically** into g ($f \sqsubseteq g$) if there exist two topological embeddings $\varphi : X_f \to X_g$ and $\psi : Y_f \to Y_g$ such that $\psi \circ f = g \circ \varphi$.

$$\begin{array}{ccc} X_g & \xrightarrow{g} & Y_g \\ \uparrow \varphi & & \uparrow \psi \\ X_f & \xrightarrow{f} & Y_f \end{array}$$

Theorem (Solecki Dichotomy)

Given X analytic space, Y separable metrizable and $f : X \to Y$ Borel function; then either $f \in \mathrm{dec}(\boldsymbol{\Sigma}^0_1)$ or $P \sqsubseteq f$.

Different reducibilities: (strong) continuous reducibility

Definition

Given X_f, Y_f, X_g, Y_g topological spaces and functions $f : X_f \to Y_f$ and $g : X_g \to Y_g$, we say that f is **continuously reducible** to g ($f \leqslant_s g$) if there exist two partial continuous functions $\varphi : X_f \rightharpoonup X_g$ and $\psi : Y_g \rightharpoonup Y_f$ such that $\forall\, x \in X_f(f(x) = \psi(g(\varphi(x))))$.

$$\begin{array}{ccc} X_g & \xrightarrow{g} & Y_g \\ \uparrow\varphi & & \downarrow\psi \\ X_f & \xrightarrow{f} & Y_f \end{array}$$

The Solecki Dichotomy can be "weakened" with $\leqslant_s$ in place of $\sqsubseteq$ and is not difficult to prove that $J \equiv_s P$. Moreover, Marks and Montalbàn recently announced:

Theorem (Generalized Solecki Dichotomy)

Given $1 \leqslant \alpha < \omega_1$, and $f : \omega^\omega \to \omega^\omega$ Borel; then either $f \in \mathrm{dec}(\mathbf{\Sigma}^0_{<(1+\alpha)})$ or $J^{(\alpha)} \leqslant_s f$.

Different reducibilities: weak continuous reducibility

Definition

Given X_f, Y_f, X_g, Y_g topological spaces and functions $f : X_f \to Y_f$ and $g : X_g \to Y_g$, we say that f is **weakly (continuously) reducible** to g ($f \leqslant_w g$) if there exist two partial continuous functions $\varphi : X_f \rightharpoonup X_g$ and $\psi : Y_g \times X_f \rightharpoonup Y_f$ such that $\forall\, x \in X_f (f(x) = \psi(g(\varphi(x)), x))$.

Again, the Solecki Dichotomy can be restated with $\leqslant_w$ in place of $\sqsubseteq$. We can now state our main result:

Theorem (Lutz, Carroy, Nicolosi)

Given X Polish space, Y separable metrizable and $f : X \to Y$ Borel; then either $f \in \mathrm{dec}(\mathbf{\Sigma}^0_n)$ or $J^{(n)} \leqslant_w f$.

This result was originally proved by Lutz for functions on the Baire space.

How to define effectivity in Polish spaces?

There are two main approaches for using effectivity in Polish spaces:

Using *recursively presented metric spaces*, by fixing:

- a compatible metric
- a countable dense sequence

Using *basic spaces*, by fixing:

- a countable basis (basic and recursive spaces of Alain Louveau).

Definition

Let (X, d) be a separable metric space and $\boldsymbol{r} = (r_i)_{i \in \omega}$ an enumeration (possibly with repetitions) of a dense subset of X. We say that $\boldsymbol{r}$ is a **recursive presentation** of X if the relations on ω^3

$$P(i, j, k) \Leftrightarrow \mathrm{d}(r_i, r_j) \leqslant q_k$$
$$Q(i, j, k) \Leftrightarrow \mathrm{d}(r_i, r_j) < q_k$$

are recursive. The structure $(X, \mathrm{d}, \boldsymbol{r})$ is called **recursively presented metric space**. If moreover (X, d) is complete, then $(X, \mathrm{d}, \boldsymbol{r})$ is called **recursively presented Polish space**.

Basic spaces

Definition

A **basic space** $\mathcal{X}$ is a pair $(X, (V_n)_{n\in\omega})$ where X is a second countable topological space, $(V_n)_{n\in\omega}$ is an enumeration (possibly with repetitions) of a countable basis of the topology of X[a], and there is a semirecursive relation $R \subseteq \omega^3$ such that:

$$V_m \cap V_n = \bigcup_{p\in\omega} \{V_p \mid R(m, n, p)\}$$

[a] The V_ns are not necessarily not empty.

The space ω is basic with the enumeration of the basis given by $V_n^\omega = \{n\}$.

Fact

Subspaces, finite products and countable products of basic spaces are basic spaces.

In particular, we have a canonical structure of basic space for each product space $X \times \omega$, with $\mathcal{X} = (X, (V_n^{\mathcal{X}})_{n\in\omega})$ basic space.

Σ^0_1 sets

Definition

A subset A of a basic space $\mathcal{X}$ is called $\Sigma^0_1(\mathcal{X})$ (also said **effectively open** in $\mathcal{X}$) if there is a semirecursive set A^* in ω such that

$$A = \bigcup_{n \in A^*} V^{\mathcal{X}}_n$$

The Σ^0_1 sets of the basic space $(\omega, (\{n\})_{n\in\omega})$, are exactly the semirecursive sets.

Definition

Given $\mathcal{X}$, $\mathcal{Y}$ basic spaces, a function $f : \mathcal{X} \to \mathcal{Y}$ is Σ^0_1-**recursive** if its diagram is Σ^0_1, that is:

$$D_f = \{(x, n) \mid f(x) \in V^{\mathcal{Y}}_n\} \in \Sigma^0_1(\mathcal{X} \times \omega)$$

Any Σ^0_1-recursive function is also continuous. Moreover, the Σ^0_1-recursive functions generalize not only the computable functions on the natural numbers but also the computable functions on ω^ω (Type-2 theory of effectivity).

Recursive spaces

Definition

Given $\mathcal{X}$, $\mathcal{Y}$ basic spaces, $f : \mathcal{X} \to \mathcal{Y}$ is a **recursive isomorphism** if it is Σ^0_1-recursive, bijective, and has Σ^0_1-recursive inverse.

Fact

Any recursively presented metric space $(X, \mathrm{d}, \boldsymbol{r})$ admits a structure of basic space by considering $V_n = \{x \in X \mid \mathrm{d}(\boldsymbol{r}_{(n)_0}, x) < q_{(n)_1}\}$.

Definition

A basic space $\mathcal{X}$ is **recursive** if it is recursively isomorphic to a subspace of a recursively presented metric space.

Proposition

Any recursive space is recursively isomorphic to a subspace of the Hilbert cube $[0,1]^\omega$.

Universal sets and relativization

Definition

Given $\mathcal{X}$ and $\mathcal{Y}$ basic spaces, $G \in \Sigma^0_1(\mathcal{X} \times \mathcal{Y})$ is **universal** for $\Sigma^0_1(\mathcal{Y})$ if

$$\forall\, P \subseteq Y(P \in \Sigma^0_1 \Leftrightarrow\, \exists\, x \in X(P = G_x))$$

where $G_x = \{y \in Y \mid G(x, y)\}$ is called x-section of G.

A similar definition can be given for $\boldsymbol{\Sigma}^0_1$ sets (i.e. open sets) in topological spaces.

Remark

Considering a α-semirecursive set instead of a semirecursive A^* in the definition of effective open set, we obtain the pointclass of α-effectively open sets that we denote with $\Sigma^{0,\alpha}_1$.

One can prove that for any recursive space $\mathcal{X}$: $\boldsymbol{\Sigma}^0_1(X) = \bigcup_{\alpha \in \omega^\omega} \Sigma^{0,\alpha}_1(\mathcal{X})$.

Lightface pointclasses and universal sets

A *lightface pointclass* is a collection of subsets of basic spaces closed under Σ^0_1-recursive preimages. The pointclasses in the **Arithmetical Hierarchy** are defined (by induction) as:

$$\Sigma^0_1 \qquad \Pi^0_1 = \neg\Sigma^0_1 \qquad \Sigma^0_{n+1} = \exists^0\Pi^0_n \qquad \Pi^0_{n+1} = \neg\Sigma^0_{n+1} \qquad \Delta^0_n = \Sigma^0_n \cap \Pi^0_n$$

and similarly one can define the **Analytical Hierarchy**:

$$\Sigma^1_1 = \exists^1\Pi^0_1 \qquad \Pi^1_1 = \neg\Sigma^1_1 \qquad \Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$$

As for Σ^0_1 sets there is a notion of universal sets, in particular the following holds:

Theorem (Universal sets for lightface pointclasses)

Given $\mathcal{X}$ recursive space and a lightface pointclass Γ between Σ^0_n, Π^0_n, Σ^1_1 and Π^1_1, then there is a set $G \in \Gamma(\omega \times \mathcal{X})$ which is universal for $\Gamma(\mathcal{X})$.

Moreover, the process of relativization can be extended also to these pointclasses, obtaining this way the corresponding *topological/boldface pointclasses*.

A particular universal system

Given an oracle $\alpha \in \omega^\omega$ we denote with W_e^α the e-th α-semirecursive set of ω (that is $W_e^\alpha = \{n \in \omega \mid \varphi_e^\alpha(n) \downarrow\}$). Given a recursive space $\mathcal{Y}$, we define:

$$H^{\mathcal{Y}}_{\Sigma^0_1} = \{(\alpha, e, y) \in \omega^\omega \times \omega \times Y \mid \exists\, n \in W_e^\alpha (y \in V_n^{\mathcal{Y}})\}$$
$$H^{\mathcal{Y}}_{\Sigma^0_{n+1}} = \{(\alpha, e, y) \in \omega^\omega \times \omega \times Y \mid \exists\, i \in \omega \neg H^{\omega \times \mathcal{Y}}_{\Sigma^0_n}(\alpha, e, i, y)\}$$

The α-section $H^{\mathcal{Y}}_{\Sigma^0_n, \alpha}$ is universal for $\Sigma^{0,\alpha}_n(\mathcal{Y})$, hence $(H^{\mathcal{Y}}_{\Sigma^0_n})_{\mathcal{Y}}$ is a *parametrization system*. The reason why we are interested in this specific parametrization system is because it has the following property:

Definition

A parametrization system $(G^{\mathcal{Y}})_{\mathcal{Y}}$ is **effectively good** if for every $k \in \omega$, and every recursive space $\mathcal{Y}$ exists a computable function $S : \omega^{k+1} \to \omega$ such that:

$$\forall\, e \in \omega\, \forall\, x \in \omega^k\, \forall\, y \in Y (G^{\omega^k \times \mathcal{Y}}(e, x, y) \Leftrightarrow G^{\mathcal{Y}}(S(e, x), y))$$

As any separable metrizable has a structure of ε-recursive space (for a strong enough oracle $\varepsilon \in \omega^\omega$), we extend this construction to any separable metrizable space.

Continuous degrees

Definition

Given $\mathcal{X}$ and $\mathcal{Y}$ recursive spaces, $y \in Y$ is **representation reducible** to $x \in X$ (and we write $y \leqslant_M x$) if there is some $e \in \omega$ such that $\Phi_e^{\mathcal{X},\mathcal{Y}}(x) = y$, where $\Phi_e^{\mathcal{X},\mathcal{Y}}$ is the e-th partial Σ_1^0-recursive function on its domain.

$\leqslant_M$ is a quasi-order (reflexive and transitive).

Definition (Continuous degrees [Mil04] and [GKN20])

Given $\mathcal{X}$ recursive space, the **continuous degree** of $x \in \mathcal{X}$ is its equivalence class under the relation $\equiv_M$ (over elements of recursive spaces).

Fact

Given $\mathcal{X}$ recursive space,

1. $\forall\, x \in X\ \exists\, z \in [0,1]^\omega (x \equiv_M z)$.
2. $\forall\, x, y \in \omega^\omega (y \leqslant_T x \Leftrightarrow y \leqslant_M x)$
3. $\forall\, x \in X\ \forall\, z \in \omega^\omega (x \leqslant_M z \Leftrightarrow\ \exists\, p \in \omega^\omega (\rho_{\mathcal{X}}(p) = x \wedge p \leqslant_T z))$

where $\rho_{\mathcal{X}} : \omega^\omega \to X$ is the admissible representation *defined as follows:*

$$\rho_{\mathcal{X}}(p) = x \Leftrightarrow \operatorname{ran}(p) = \{n \in \omega \mid x \in V_n^{\mathcal{X}}\}$$

The Turing jump for recursive spaces

The notion of Turing Jump can be extended to all recursive spaces

Definition ([GKN20])

Given $\mathcal{X}$ recursive space, the $\Sigma^{0,\alpha}_n$**-jump** is the function defined as:

$$J^{(n),\alpha}_{\mathcal{X}} : \mathcal{X} \to 2^\omega$$
$$x \mapsto J^{(n),\alpha}_{\mathcal{X}}(x) = \{e \in \omega \mid x \in H^{\mathcal{X}}_{\Sigma^0_n,\alpha,e}\}$$

Similarly, we define the $\Sigma^{0,\alpha}_n$-jump for ε-recursive spaces and $\alpha \geqslant_T \varepsilon$.
The usual Turing Jump on the Baire space is "equivalent" to the Σ^0_1-jump $J^{(1),\varnothing}_{\omega^\omega}$. In fact:

$$\forall\, x \in \omega^\omega (J^{(1),\varnothing}_{\omega^\omega}(x) \equiv_T J(x))$$

For this reason, we simply write $x^{(n)}$ instead of $J^{(n-1)} \circ J^{(1),\varnothing}_{\mathcal{X}}(x)$.

The defined operator shares common property with the usual Turing Jump:

Proposition

Given $\mathcal{X}$ and $\mathcal{Y}$ recursive spaces, then:

- $\forall\, x \in X\, \forall\, y \in Y (x \leqslant_M y \Rightarrow J^{(1),\varnothing}_{\mathcal{X}}(x) \leqslant_1 J^{(1),\varnothing}_{\mathcal{Y}}(y))$
- $\forall\, x \in X\, \exists\, p \in \omega^\omega (\rho_{\mathcal{X}}(p) = x \wedge J^{(1),\varnothing}_{\mathcal{X}}(x) \equiv_1 p')$

Moreover, it is universal in the following sense:

Proposition (Folklore?)

Given X, Y separable metrizable and $f : X \to Y$ $\mathbf{\Sigma}^0_{n+1}$-measurable then $f \leqslant_s J^{(n),\varnothing}_{\mathbb{R}^\omega}$.

Proof.

Let $i : X \to \mathbb{R}^\omega$ be a topological embedding, then $f \circ i^{-1}$ is $\Sigma^{0,\gamma}_{n+1}$-recursive on its domain $\mathcal{Z}$ for some powerful enough oracle $\gamma \in \omega^\omega$.Define
$\tau(z) = \bigcap_{m\in\omega}\{V^{\mathcal{Y}}_m \mid \exists\, i \in \omega z(S(e,i,m)) = 0 \wedge \operatorname{diam}(V^{\mathcal{Y}}_m) < 2^{-(m)_0}\}$, then $f = \tau \circ J^{(n),\gamma}_{\mathcal{X}} \circ i$:

$$f \circ i^{-1}(x) \in V^{\mathcal{Y}}_m \wedge \operatorname{diam}(V^{\mathcal{Y}}_m) < 2^{-(m)_0} \Leftrightarrow (x, m) \in H^{\mathcal{Z}\times\omega}_{\Sigma^0_{n+1},\gamma,e} \wedge \operatorname{diam}(V^{\mathcal{Y}}_m) < 2^{-(m)_0}$$

$$\Leftrightarrow \exists\, i \in \omega\ (x, m) \notin H^{\mathcal{Z}\times\omega^2}_{\Sigma^0_n,\gamma,e,i} \wedge \operatorname{diam}(V^{\mathcal{Y}}_m) < 2^{-(m)_0}$$

$$\Leftrightarrow \exists\, i \in \omega\, x \notin H^{\mathcal{Z}}_{\Sigma^0_n,\gamma,S(e,i,m)} \wedge \operatorname{diam}(V^{\mathcal{Y}}_m) < 2^{-(m)_0}$$

$$\Leftrightarrow \exists\, i \in \omega\ J^{(n),\gamma}_{\mathcal{Z}}(x)(S(e,i,m)) = 0 \wedge \operatorname{diam}(V^{\mathcal{Y}}_m) < 2^{-(m)_0} \Rightarrow \tau \circ J^{(n),\gamma}_{\mathcal{Z}}(x) \in V^{\mathcal{Y}}_m$$

observe that τ is partial continuous as defined using a Cantor scheme. $\square$

Shore-Slaman Join Theorem

Theorem (Shore-Slaman Join Theorem for recursive spaces [GKN20])

Let $\mathcal{X}$ and $\mathcal{Y}$ be recursive spaces, $x \in X$, $y \in Y$, and $n \in \omega$. If $y \not\leqslant_M x^{(n)}$, then there is a $G \in 2^\omega$ such that $G \geqslant_M x \wedge G^{(n+1)} \equiv_M G \oplus y$.
Moreover, the same result holds for any ordinal $\xi < \omega_1^{CK}$, in particular: if $\forall\, \zeta < \xi (y \not\leqslant_M x^{(\zeta)})$, then $\exists\, G \in 2^\omega (G \geqslant_M x \wedge G^{(\xi)} \equiv_M G \oplus y)$.

Theorem (Generalized Posner-Robinson for recursive spaces)

Given $\mathcal{X}$ and $\mathcal{Y}$ recursive spaces, for every $n \in \omega$
$\forall\, x \in X\ \forall\, y \in Y (y \leqslant_M x^{(n)} \ \dot\vee\ \exists\, g \in 2^\omega (x \oplus y \oplus g \geqslant_M (g \oplus x)^{(n+1)}))$.

A modification of Lutz's game

Given two functions $f : \omega^\omega \to \omega^\omega$ and $g : \mathcal{X} \to \mathcal{Y}$ (where $\mathcal{X}$ and $\mathcal{Y}$ are recursive spaces) we define the game $G_M(f, g)$ as follows: Player 2 first plays a code $e \in \omega$ corresponding to a Σ^0_1-recursive function. For the rest of the game, Player 1 plays each turn a digit of a real $x = x_0 x_1 \ldots \in \omega^\omega$ and Player 2 plays a digit of two reals, $b = b_0 b_1 \ldots \in \omega^\omega$ and $z = z_0 z_1 \ldots \in \omega^\omega$.

Player 1		x_0	x_1	$\ldots$
Player 2	e	$V^{\mathcal{X}}_{b_0}$, z_0	$V^{\mathcal{X}}_{b_1}$, z_1	$\ldots$

Player 2 wins if and only if b is a $\rho_{\mathcal{X}}$-name for an element $y \in \mathcal{X}_g$ (i.e. $b \in \mathrm{dom}(\rho_{\mathcal{X}})$) and $f(x) = \Phi_e^{(\mathcal{Y} \times \omega^\omega \times \omega^\omega), \omega^\omega}(g(\rho_{\mathcal{X}}(b)), x, z)$.

Remark

If f and g are Borel functions and the domain of the representation $\rho_{\mathcal{X}} : \omega^\omega \rightharpoonup \mathcal{X}$ is Borel, then the payoff of the game $G_M(f, g)$ is Borel.

Lemma (Lutz, Carroy, Nicolosi)

- *If Player 2 has a winning strategy for $G_M(f,g)$ then $f \leqslant_w g$.*
- *If Player 1 has a winning strategy for $G_M(J^{(n)},g)$ then $g \in \mathrm{dec}(\boldsymbol{\Sigma}^0_n)$.*

Sketch of the proof.

- Consider as $\varphi(x) = \rho_{\mathcal{X}} \circ \phi(x)$ (where ϕ is the function determined by the winning strategy of Player 2) and as $\psi(w,x) = \Phi_e^{(\mathcal{Y}\times 2^\omega \times \omega^\omega),\omega^\omega}(w,x,z)$.
- We consider the function $\tilde{g} = g \circ \rho_{\mathcal{X}}$. One can prove that:
  1. $\tilde{g}$ decomposable in $\boldsymbol{\Sigma}^0_n$-measurable functions on $\rho_{\mathcal{X}}$-saturated pieces $\Rightarrow g \in \mathrm{dec}(\boldsymbol{\Sigma}^0_n)$: If $\{B_m\}_{m\in\omega}$ is such a covering, then $\{\rho_{\mathcal{X}}[B_m]\}_{m\in\omega}$ is a covering such that on each piece g is $\boldsymbol{\Sigma}^0_n$-measurable. Indeed, as $\rho_{\mathcal{X}} : \omega^\omega \rightharpoonup X$ is continuous, open, with Polish fibers, we can use a theorem of Saint-Raymond to prove that, given an open set U, then $(g\restriction\rho_{\mathcal{X}}[B_m])^{-1}[U] = \rho_{\mathcal{X}}[(\tilde{g}\restriction B_m)^{-1}[U]] \in \boldsymbol{\Sigma}^0_n$.
  2. If for some $\tau \in \omega^\omega$ $\forall\, b \in \mathrm{dom}(\tilde{g})(\tilde{g}(b) \leqslant_M (\rho_{\mathcal{X}_g}(b) \oplus \tau)^{(n-1)})$, then $\tilde{g}$ is decomposable in $\boldsymbol{\Sigma}^0_n$-measurable functions on the following $\rho_{\mathcal{X}_g}$-saturated domains:
  $$B_m = \{b \in \mathrm{dom}(\rho_{\mathcal{X}_g}) \mid \tilde{g}(b) = \Phi_m^{(\mathcal{X}_g\times\omega^\omega),\mathcal{Y}_g}((\rho_{\mathcal{X}_g}(b)\oplus\tau)^{(n-1)})\}$$

Remains to prove that $\forall\, b \in \mathrm{dom}(\tilde{g})(\tilde{g}(b) \leqslant_M \rho_{\mathcal{X}}(b) \oplus \tau)$ where τ is the winning strategy for Player 1.

Sketch of the proof. (continued).

Towards a contradiction, suppose that for some $b \in \mathrm{dom}(\tilde{g})$ $\tilde{g}(b) \nleqslant_M \rho_{\mathcal{X}}(b) \oplus \tau$.
Then, there is a $w \in \omega^\omega$ such that:

$$\tilde{g}(b) \nleqslant_M w \wedge (\rho_{\mathcal{X}}(b) \oplus \tau)^{(n-1)} \leqslant_M w$$

and, by Generalized Posner-Robinson, there is a $v \in \omega^\omega$: $\tilde{g}(b) \oplus w \oplus v \geqslant_M (v \oplus w)^{(n)}$.Now we explain how to win while playing as Player 2 against τ, actually to reach the (stronger) condition $\Phi_e^{\mathcal{Y}_g \times \omega^\omega, \omega^\omega}(\tilde{g}(b), z) = J^{(n)}(x)$: Observe that differently from the original proof, $v \oplus w$ could not compute Player 1 moves because it could not compute the other element b. So we have to find a different name p (using properties of representation reducibility) such that $w \geqslant_M p$. Then if Player 2 plays $b = p$ and $z = v \oplus w$, then:

$$\tilde{g}(p) \oplus z \equiv_M \tilde{g}(p) \oplus w \oplus v \geqslant_M (w \oplus v)^{(n)} \geqslant_M x^{(n)}$$

In particular, this computation depends uniformly on the first element played e, that is:

$$\forall\ e \in \omega \left(\Phi_a^{\mathcal{Y}_g \times \omega^\omega \times \omega, \omega^\omega}(\tilde{g}(p), z, e) = J_n(x) \right)$$

Therefore, using Kleene's Recursion Theorem for recursive spaces, one get the right e to play. $\lightning$ $\square$

Using Borel determinacy we get

Theorem (Lutz, Carroy, Nicolosi)

Given $\mathcal{X}$, $\mathcal{Y}$ recursive spaces such that $\rho_{\mathcal{X}}$ has Borel domain and $g : \mathcal{X} \to \mathcal{Y}$ Borel function; then either $g \in \mathrm{dec}(\boldsymbol{\Sigma}^0_n)$ or $J^{(n)} \leqslant_w g$.

Clearly, under the Axiom of Determinacy, this result generalizes to all recursive spaces and functions. In addition:

Fact

The domain of the considered representation is Borel for any recursively presented Polish spaces.

Thus we get:

Corollary

Given $\mathcal{X}$ recursively presented Polish space, $\mathcal{Y}$ recursive space and $g : X \to Y$ Borel function; then either $g \in \mathrm{dec}(\boldsymbol{\Sigma}^0_n)$ or $J^{(n)} \leqslant_w g$.

Further directions

- Is it possible to find a game that characterizes weak reducibility $\leqslant_w$ in a wider context? (e.g. in separable metrizable spaces)
- How to extend this result to all Borel hierarchy?
- Can this result be strengthened up to the Generalized Solecki Dichotomy proved by Marks and Montalbàn? What about the topological embedding?

Thank you for your attention.

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