A game characterization for weak continuous reducibility

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The original problem

Question (Luzin (early 1900s))

Is every Borel function between Polish spaces decomposable as a countable union of continuous functions?

This question has been answered negatively by:

1. Novikov
2. Keldysh (1934), who gave a level-by-level generalization: for each $1 \leqslant \alpha < \omega_1$ a function $\boldsymbol{\Sigma}^0_{\alpha+1}$-measurable which cannot be decomposed as a countable union of functions of lower complexity.

Notation

We write $f \in \mathrm{dec}(\boldsymbol{\Sigma}^0_\alpha)$ *if* f *is decomposable as a countable union of* $\boldsymbol{\Sigma}^0_\alpha$*-measurable functions (on their domains).*

The Solecki Dichotomy

Although such decomposability of Borel functions is false in general, the non-decomposable functions can be characterized as follows

Theorem (Solecki Dichotomy)

Given X analytic space, Y separable metrizable and $f : X \to Y$ Borel; then either $f \in \mathrm{dec}(\boldsymbol{\Sigma}^0_1)$ or f "contains" the Pawlikowski *function $P : (\omega+1)^\omega \to \omega^\omega$, defined as:*

$$P(\eta)(n) = \begin{cases} 0 & \textit{if } \eta(n) = \omega \\ \eta(n)+1 & \textit{otherwise} \end{cases}$$

This result was first proved by Solecki (1998) for Baire class 1 functions and then extended to all Borel functions by Zapletal (2004) and by Pawlikowski and Sabok to all functions with analytic graphs (2012).

Fact (Carroy, Lutz)

The Pawlikowski function is "equivalent" to the **Turing Jump**, *i.e.,*

$$\begin{aligned} J : \omega^\omega &\to 2^\omega \\ x &\mapsto J(x) = x' = \{e \in \omega \mid \varphi^x_e(e)\downarrow\} \end{aligned}$$

The Solecki Dichotomy (more precisely)

Definition

Given X_f, Y_f, X_g, Y_g topological spaces and functions $f : X_f \to Y_f$ and $g : X_g \to Y_g$, we say that f **embeds topologically** into g ($f \sqsubseteq g$) if there exist two topological embeddings $\varphi : X_f \to X_g$ and $\psi : Y_f \to Y_g$ such that $\psi \circ f = g \circ \varphi$.

$$\begin{array}{ccc} X_g & \xrightarrow{g} & Y_g \\ \uparrow \varphi & & \uparrow \psi \\ X_f & \xrightarrow{f} & Y_f \end{array}$$

Theorem (Solecki Dichotomy)

Given X analytic space, Y separable metrizable and $f : X \to Y$ Borel function; then either $f \in \mathrm{dec}(\mathbf{\Sigma}^0_1)$ or $P \sqsubseteq f$.

Different reducibilities: (strong) continuous reducibility

Definition

Given X_f, Y_f, X_g, Y_g topological spaces and functions $f : X_f \to Y_f$ and $g : X_g \to Y_g$, we say that f is **continuously reducible** to g ($f \leqslant_s g$) if there exist two partial continuous functions $\varphi : X_f \rightharpoonup X_g$ and $\psi : Y_g \rightharpoonup Y_f$ such that $\forall\, x \in X_f(f(x) = \psi(g(\varphi(x))))$.

$$\begin{array}{ccc} X_g & \xrightarrow{g} & Y_g \\ \uparrow \varphi & & \downarrow \psi \\ X_f & \xrightarrow{f} & Y_f \end{array}$$

The Solecki Dichotomy can be "weakened" by replacing $\leqslant_s$ with $\sqsubseteq$ and it is not difficult to prove that $J \equiv_s P$. Moreover, Marks and Montalbán recently announced:

Theorem (Generalized Solecki Dichotomy)

Given $1 \leqslant \alpha < \omega_1$, and $f : \omega^\omega \to \omega^\omega$ Borel; then either $f \in \mathrm{dec}(\boldsymbol{\Sigma}^0_{<(1+\alpha)})$ or $J^{(\alpha)} \leqslant_s f$.

Different reducibilities: weak continuous reducibility

Definition

Given X_f, Y_f, X_g, Y_g topological spaces and functions $f : X_f \to Y_f$ and $g : X_g \to Y_g$, we say that f is **weakly (continuously) reducible** to g ($f \leqslant_w g$) if there exist two partial continuous functions $\varphi : X_f \rightharpoonup X_g$ and $\psi : Y_g \times X_f \rightharpoonup Y_f$ such that

$$\forall\, x \in X_f (f(x) = \psi(g(\varphi(x)), x)).$$

Again, the Solecki Dichotomy can be restated with $\leqslant_w$ in place of $\sqsubseteq$. We can now state our main result:

Theorem (Lutz, Carroy, Nicolosi)

For every $n \geqslant 1$, given X standard Borel, Y separable metrizable and $f : X \to Y$ Borel; then either $f \in \mathrm{dec}(\boldsymbol{\Sigma}^0_n)$ or $J_n \leqslant_w f$.

This result was originally proved by Lutz for functions on the Baire space.

How to define effectivity in Polish spaces?

Definition

Let (X, d) be a separable metric space and $\boldsymbol{r} = (r_i)_{i\in\omega}$ an enumeration (possibly with repetitions) of a dense subset of X. We say that $\boldsymbol{r}$ is a **recursive presentation** of X if the relations on ω^3

$$P(i,j,k) \Leftrightarrow \mathrm{d}(r_i, r_j) \leqslant q_k \qquad Q(i,j,k) \Leftrightarrow \mathrm{d}(r_i, r_j) < q_k$$

are computable. The structure $\mathcal{X} = (X, \mathrm{d}, \boldsymbol{r})$ is called **recursively presented metric space**. If moreover (X, d) is complete, then $\mathcal{X}$ is called **recursively presented Polish space**.

For any recursively presented metric space we fix the following enumeration of the basis:

$$V_i^{\mathcal{X}} = B_{q_{(i)_1}}(\boldsymbol{r}_{(i)_0}) = \{x \in X \mid d(x, \boldsymbol{r}_{(i)_0}) < q_{(i)_1}\}$$

where $(q_l)_{l\in\omega}$ is a fixed computable enumeration of $\mathbb{Q}^+$.

Remark

Having fixed a basis, we can associate to $\mathcal{X}$ the canonical neighborhood representation $\rho_{\mathcal{X}} : \omega^\omega \rightharpoonup \mathcal{X}$ *defined by:*

$$\rho_{\mathcal{X}}(p) = x \Leftrightarrow \mathrm{ran}(p) = \mathrm{N}_{\mathrm{base}}(x) \coloneqq \{n \in \omega \mid x \in V_n^{\mathcal{X}}\}$$

Σ^0_1 sets and Σ^0_1-recursive functions

Definition

A subset A of a r.p.m.s. $\mathcal{X}$ is called $\Sigma^0_1(\mathcal{X})$ (also called **effectively open** in $\mathcal{X}$) if there is a c.e. set A^* in ω such that

$$A = \bigcup_{n \in A^*} V_n^{\mathcal{X}}$$

The Σ^0_1 sets of the r.p.m.s. $(\omega, (\{n\})_{n\in\omega})$ are exactly the c.e. sets.

Definition

Given $\mathcal{X}$, $\mathcal{Y}$ r.p.m.s., a function $f : \mathcal{X} \to \mathcal{Y}$ is Σ^0_1-**recursive** if its diagram is Σ^0_1, that is:

$$D_f = \{(x, n) \mid f(x) \in V_n^{\mathcal{Y}}\} \in \Sigma^0_1(\mathcal{X} \times \omega)$$

Remark

- *Any Σ^0_1-recursive function is also continuous.*
- *Σ^0_1-recursive functions generalize not only the computable functions on the natural numbers but also the computable functions on ω^ω (Type-2 theory of effectivity).*
- *The Σ^0_1-recursive functions coincide with the $(\rho_{\mathcal{X}}, \rho_{\mathcal{Y}})$-computable functions induced by the canonical neighborhood representation.*

Continuous degrees

Definition

Given $\mathcal{X}$ and $\mathcal{Y}$ r.p.m.s., $y \in Y$ is **representation reducible** to $x \in X$ (and we write $y \leqslant_M x$) if there is some $e \in \omega$ such that $\Phi_e^{\mathcal{X},\mathcal{Y}}(x) = y$, where $\Phi_e^{\mathcal{X},\mathcal{Y}}$ is the e-th partial Σ_1^0-recursive function on its domain.

$\leqslant_M$ is a quasi-order (reflexive and transitive).

Definition (Continuous degrees [Mil04] and [GKN21])

Given $\mathcal{X}$ r.p.m.s., the **continuous degree** of $x \in \mathcal{X}$ is its equivalence class under the relation $\equiv_M$ (over elements of recursive spaces).

Fact

Given $\mathcal{X}$, $\mathcal{Y}$ r.p.m.s.,

1. $\forall\, x \in X\ \forall\, y \in Y (y \leqslant_M x \Leftrightarrow \mathrm{N}_{\mathrm{base}}(y) \leqslant_e \mathrm{N}_{\mathrm{base}}(x))$
2. $\forall\, x, y \in \omega^\omega (y \leqslant_T x \Leftrightarrow y \leqslant_M x)$

Universal sets

Definition

Given $\mathcal{X}$ and $\mathcal{Y}$ r.p.m.s., $G \in \Sigma^0_1(\mathcal{X} \times \mathcal{Y})$ is **universal** for $\Sigma^0_1(\mathcal{Y})$ if

$$\forall\, P \subseteq Y (P \in \Sigma^0_1 \Leftrightarrow\, \exists\, x \in X (P = G_x))$$

where $G_x = \{y \in Y \mid G(x, y)\}$ is called x-section of G.

Definition

A **universal system** for Σ^0_1 sets is a class-function $\mathcal{Y} \mapsto G^{\mathcal{Y}} \subseteq \omega \times Y$ such that $G^{\mathcal{Y}}$ is universal for every r.p.m.s. $\mathcal{Y}$.

A particular universal system

Denote by $W_e = \{n \in \omega \mid \varphi_e(n)\downarrow\}$ the e-th c.e. set of ω. Given a r.p.m.s. $\mathcal{Y}$, we define:

$$H^{\mathcal{Y}}_{\Sigma^0_1} = \{(e, y) \in \omega \times Y \mid \exists\, n \in W_e(y \in V^{\mathcal{Y}}_n)\}$$

Each $H^{\mathcal{Y}}_{\Sigma^0_1}$ is universal for $\Sigma^0_1(\mathcal{Y})$, hence $(H^{\mathcal{Y}}_{\Sigma^0_1})_{\mathcal{Y}}$ is an *universal system*.
The reason why we are interested in this specific universal system is because it has the following property, which is basically the *s-m-n theorem*:

Definition

A universal system $(G^{\mathcal{Y}})_{\mathcal{Y}}$ is **effectively good** if for every $k \in \omega$, and every r.p.m.s. $\mathcal{Y}$, there exists a computable function $S : \omega^{k+1} \to \omega$ such that:

$$\forall\, e \in \omega\ \forall\, x \in \omega^k\ \forall\, y \in Y(G^{\omega^k \times \mathcal{Y}}(e, x, y) \Leftrightarrow G^{\mathcal{Y}}(S(e, x), y))$$

Since every separable metrizable space admits a structure of ε-r.p.m.s. (for a strong enough oracle $\varepsilon \in \omega^\omega$), we can extend this construction to any separable metrizable space.

The Turing jump on Polish spaces

The notion of Turing Jump can be extended to recursively presented metric spaces:

Definition ([GKN21])

Given $\mathcal{X}$ r.p.m.s., the Σ^0_1**-jump** is the function defined as:

$$J^{(1)}_{\mathcal{X}} : \mathcal{X} \to 2^\omega$$
$$x \mapsto J^{(1)}_{\mathcal{X}}(x) = \{e \in \omega \mid x \in H^{\mathcal{X}}_{\Sigma^0_1, e}\}$$

Similarly, we can define the Σ^0_1-jump for ε-r.p.m.s..
The usual Turing Jump on the Baire space is "equivalent" to the Σ^0_1-jump $J^{(1)}_{\omega^\omega}$. In fact:

$$\forall\, x \in \omega^\omega (J^{(1)}_{\omega^\omega}(x) \equiv_T J(x))$$

For this reason, we simply write $x^{(n)}$ instead of $J_{n-1} \circ J^{(1)}_{\mathcal{X}}(x)$. Moreover, it satisfies:

Theorem (Generalized Posner-Robinson for recursively presented spaces)

Given $\mathcal{X}$ and $\mathcal{Y}$ r.p.m.s., for every $n \in \omega$

$$\forall\, x \in X\, \forall\, y \in Y \left(y \leqslant_M x^{(n)} \;\dot{\vee}\; \exists\, g \in 2^\omega (x \oplus y \oplus g \geqslant_M (g \oplus x)^{(n+1)}) \right).$$

A modification of Lutz's game

Given two functions $f : \omega^\omega \to \omega^\omega$ and $g : \mathcal{X} \to \mathcal{Y}$ (where $\mathcal{X}$ and $\mathcal{Y}$ are r.p.m.s.) we define the game $G_M(f, g)$ as follows: Player 2 first plays a code $e \in \omega$ corresponding to a Σ^0_1-recursive function. For the rest of the game, Player 1 plays each turn one digit of a real $x = x_0 x_1 \ldots \in \omega^\omega$ and Player 2 plays one more digit of a real $z = z_0 z_1 \ldots \in \omega^\omega$ and either one or zero more digits of a real $b = b_0 b_1 \ldots \in \omega^\omega$.

Player 1		x_0		x_1		x_2	$\ldots$
Player 2	e		$b_0,\ z_0$		z_1	$b_1,\ z_2$	$\ldots$

Player 2 wins if and only if b is a $\rho_{\mathcal{X}}$-name for an element $y \in \mathcal{X}$ (i.e. $b \in \mathrm{dom}(\rho_{\mathcal{X}})$) and

$$f(x) = \Phi_e^{(\mathcal{Y} \times \omega^\omega \times \omega^\omega), \omega^\omega}(g(\rho_{\mathcal{X}}(b)), x, z).$$

Remark

If f and g are Borel functions and the domain of the representation $\rho_{\mathcal{X}} : \omega^\omega \rightharpoonup \mathcal{X}$ is Borel, then the payoff of the game $G_M(f, g)$ is Borel.

Continuity on the Baire space

Proposition

A function $f : \omega^\omega \to \omega^\omega$ is continuous *if and only if there is an $F : \omega^{<\omega} \to \omega^{<\omega}$ such that:*

1. *F is* monotone*: $\forall\, s, t \in \omega^{<\omega}(s \preceq t \Rightarrow F(s) \preceq F(t))$*
2. *F is an* approximating function *for f: $F(x) = y \Leftrightarrow \forall\, n \in \omega\ \exists\, m \geqslant n(y{\upharpoonright}n \preceq F(x){\upharpoonright}m)$*

Requiring that the approximating function is also computable one obtains the (Type-2) computable functions on ω^ω.

Remark

Considering the basis for the Baire space ω^ω consisting of sets of the form

$$N_s = \{x \in \omega^\omega \mid s \prec x\} \quad \forall\, s \in \omega^{<\omega}$$

the two conditions granting the continuity can be translated as:

1. $\forall\, s, t \in \omega^{<\omega}(s \preceq t \Rightarrow N_{F(s)} \supseteq N_{F(t)})$
2. $\forall\, x \in \omega^\omega(\mathrm{diam}(N_{F(x\upharpoonright n)}) \xrightarrow[n\to\infty]{} 0)$

Continuity from the Baire space

The same idea can be used to characterize continuity from the Baire space to any metric space.

Definition

Let X be a metric space. An ω-scheme on X is a family $\mathcal{S} = \{B_s \mid s \in \omega^{<\omega}\}$ of subsets of X such that the following conditions hold:

1. $\forall\, s, t \in \omega^{<\omega} (s \preceq t \Rightarrow B_t \subseteq B_s)$ (Monotonicity)
2. $\forall\, x \in \omega^{\omega} (\mathrm{diam}(B_{x\restriction n}) \xrightarrow[n\to\infty]{} 0)$ (Vanishing diameters)

Every ω-scheme canonically induces a (partial) function with domain

$$D_{\mathcal{S}} = \left\{x \in \omega^{\omega} \mid \bigcap_{n\in\omega} B_{x\restriction n} \neq \varnothing\right\}$$

defined as $f_{\mathcal{S}}(x) \in \bigcap_{n\in\omega} B_{x\restriction n}$.

Proposition

Given a metric space X, a function $f : \omega^{\omega} \to X$ is continuous if and only if it is induced by an ω-scheme.

This game fully characterizes the dichotomy

Lemma (Lutz, Carroy, Nicolosi)

Player 1 has a winning strategy for $G_M(J_n, g)$ if and only if $g \in \mathrm{dec}(\mathbf{\Sigma}^0_n)$.

Lemma (Lutz, Carroy, Nicolosi)

Player 2 has a winning strategy for $G_M(f, g)$ if and only if $f \leqslant_w g$.

Sketch of the proof.

- $\Rightarrow$ Set $\varphi(x) = \rho_{\mathcal{X}} \circ b(x)$ and set $\psi(w, x) = \Phi_e^{(\mathcal{Y}\times\omega^\omega\times\omega^\omega),\omega^\omega}(w, x, z(x))$ (where $b(x)$ and $z(x)$ are the continuous functions determined by the winning strategy of Player 2).
- $\Leftarrow$ Suppose that $\varphi : \omega^\omega \to \mathcal{X}$ and $\psi : \mathcal{Y} \times \omega^\omega \rightharpoonup \omega^\omega$ witness the reduction. Then, there is an $e \in \omega$ and a $z \in \omega^\omega$ such that $\forall\, y \in \mathcal{Y}\ \forall\, x \in \omega^\omega (\psi(y, x) = \Phi_e(y, x, z))$.
  As $\varphi : \omega^\omega \to \mathcal{X}$ is continuous, it is induced by the Lusin scheme $\{\varphi[N_s] \mid s \in \omega^{<\omega}\}$.
  In particular, for any $x \in \omega^\omega$:

$$\forall\, \varepsilon > 0\ \exists\, N > 0\ \forall\, n > N\, \mathrm{diam}(\phi[N_{x\restriction n}]) < \varepsilon$$

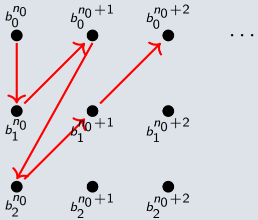
  Therefore, if Player 1 plays the digits of a real x, we let Player 2 play the chosen e and z, and wait before playing the first digit of a name for an element in $\mathcal{X}$.

Sketch of the proof. (continued).

Let n_0 be the least n such that $\varphi[N_{x\restriction n}]$ is bounded, then we can consider the real

$$b^{n_0} = b_0^{n_0} b_1^{n_0} \dots$$

whose range is the enumeration of all the balls $V_i^{\mathcal{X}}$ containing $\varphi[N_{x\restriction n_0}]$.
Similarly, we consider the reals b^m for all $m > n_0$ containing $\varphi[N_{x\restriction m}]$.
Then from the n_0-th turn, Player 2 starts playing all the digits of all the b^ms following the diagonal enumeration, as pictured below:



As any ω-scheme has the property of vanishing diameter, it follows that any ball of the fixed basis containing $\varphi(x)$ will eventually appear in the play of Player 2, and hence the resulting b is a $\rho_{\mathcal{X}}$-name for it. □

When is the game determined?

Using Borel determinacy we get

Theorem (Lutz, Carroy, Nicolosi)

For every $n \geqslant 1$, given $\mathcal{X}$, $\mathcal{Y}$ r.p.m.s. such that $\rho_{\mathcal{X}}$ has Borel domain and $g : \mathcal{X} \to \mathcal{Y}$ Borel function; then either $g \in \mathrm{dec}(\mathbf{\Sigma}^0_n)$ or $J_n \leqslant_w g$.

In addition:

Fact (Nicolosi)

For a r.p.m.s., the domain of the canonical neighborhood representation $\rho_{\mathcal{X}}$ is Δ^1_1 if and only if X is Δ^1_1 when viewed as subset of its metric completion.

In topological terms, the domain of $\rho_{\mathcal{X}}$ is Borel exactly when X is *standard Borel*.
Thus we get:

Corollary

For every $n \geqslant 1$, given X standard Borel, Y separable metrizable and $g : X \to Y$ Borel function; then either $g \in \mathrm{dec}(\mathbf{\Sigma}^0_n)$ or $J_n \leqslant_w g$.

Thank you for your attention.

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