A game characterization for weak continuous reducibility

Computability, Complexity, and Randomness 2026 - Leeds

Orazio Nicolosi
joint work with Raphaël Carroy

Dipartimento di matematica "Giuseppe Peano"

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Games for the decomposability of Borel functions

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The original problem

Question (Luzin (early 1900s))

Is every Borel function between Polish spaces decomposable as a countable union of continuous functions?

This question has been answered negatively by:

1. Novikov
2. Keldysh (1934) who gave a level-by-level generalization: for each $1 \leqslant \alpha < \omega_1$ a function $\mathbf{\Sigma}^0_{\alpha+1}$-measurable which cannot be decomposed as countable union of functions of lower complexity.

Notation

We write $f \in \mathrm{dec}(\mathbf{\Sigma}^0_\alpha)$ *if* f *is decomposable as a countable union of* $\mathbf{\Sigma}^0_\alpha$*-measurable functions (on their domains).*

The Solecki Dichotomy

Although such decomposability of Borel functions is false in general, the non-decomposable functions can be characterized as follows

Theorem (Solecki Dichotomy)

Given X analytic space, Y separable metrizable and $f : X \to Y$ Borel; then either $f \in \mathrm{dec}(\mathbf{\Sigma}^0_1)$ or f "contains" the Pawlikowski *function $P : (\omega+1)^\omega \to \omega^\omega$, defined as:*

$$P(\eta)(n) = \begin{cases} 0 & \text{if } \eta(n) = \omega \\ \eta(n)+1 & \text{otherwise} \end{cases}$$

This result was first proved by Solecki (1998) for Baire class 1 functions and then extended to all Borel functions by Zapletal (2004) and by Pawlikowski and Sabok to all functions with analytic graphs (2012).

Fact (Carroy, Lutz)

The Pawlikowski function is "equivalent" to the **Turing Jump** *i.e.,*

$$\begin{aligned} J : \omega^\omega &\to 2^\omega \\ x &\mapsto J(x) = x' = \{e \in \omega \mid \varphi_e^x(e)\downarrow\} \end{aligned}$$

The Solecki Dichotomy (more precisely)

Definition

Given X_f, Y_f, X_g, Y_g topological spaces and functions $f : X_f \to Y_f$ and $g : X_g \to Y_g$, we say that f **embeds topologically** into g ($f \sqsubseteq g$) if there exist two topological embeddings $\varphi : X_f \to X_g$ and $\psi : Y_f \to Y_g$ such that $\psi \circ f = g \circ \varphi$.

$$\begin{array}{ccc} X_g & \xrightarrow{g} & Y_g \\ \uparrow \varphi & & \uparrow \psi \\ X_f & \xrightarrow{f} & Y_f \end{array}$$

Theorem (Solecki Dichotomy)

Given X analytic space, Y separable metrizable and $f : X \to Y$ Borel function; then either $f \in \mathrm{dec}(\boldsymbol{\Sigma}^0_1)$ or $P \sqsubseteq f$.

Different reducibilities: (strong) continuous reducibility

Definition

Given X_f, Y_f, X_g, Y_g topological spaces and functions $f : X_f \to Y_f$ and $g : X_g \to Y_g$, we say that f is **continuously reducible** to g ($f \leqslant_s g$) if there exist two partial continuous functions $\varphi : X_f \rightharpoonup X_g$ and $\psi : Y_g \rightharpoonup Y_f$ such that $\forall\, x \in X_f (f(x) = \psi(g(\varphi(x))))$.

$$\begin{array}{ccc} X_g & \xrightarrow{g} & Y_g \\ \uparrow\varphi & & \downarrow\psi \\ X_f & \xrightarrow{f} & Y_f \end{array}$$

The Solecki Dichotomy can be "weakened" by replacing $\leqslant_s$ with $\sqsubseteq$ and it is not difficult to prove that $J \equiv_s P$. Moreover, Marks and Montalbán recently announced:

Theorem (Generalized Solecki Dichotomy)

Given $1 \leqslant \alpha < \omega_1$, and $f : \omega^\omega \to \omega^\omega$ Borel; then either $f \in \mathrm{dec}(\boldsymbol{\Sigma}^0_{<(1+\alpha)})$ or $J^{(\alpha)} \leqslant_s f$.

Different reducibilities: weak continuous reducibility

Definition

Given X_f, Y_f, X_g, Y_g topological spaces and functions $f : X_f \to Y_f$ and $g : X_g \to Y_g$, we say that f is **weakly (continuously) reducible** to g ($f \leqslant_w g$) if there exist two partial continuous functions $\varphi : X_f \rightharpoonup X_g$ and $\psi : Y_g \times X_f \rightharpoonup Y_f$ such that

$$\forall\, x \in X_f (f(x) = \psi(g(\varphi(x)), x)).$$

Again, the Solecki Dichotomy can be restated with $\leqslant_w$ in place of $\sqsubseteq$. We can now state our main result:

Theorem (Lutz, Carroy, Nicolosi)

For every $n \geqslant 1$, given X analytic space, Y separable metrizable and $f : X \to Y$ Borel; then either $f \in \mathrm{dec}(\boldsymbol{\Sigma}^0_n)$ or $J_n \leqslant_w f$.

This result was originally proved by Lutz for functions on the Baire space.

How to define effectivity in Polish spaces?

Definition

Let (X, d) be a separable metric space and $\boldsymbol{r} = (r_i)_{i\in\omega}$ an enumeration (possibly with repetitions) of a dense subset of X. We say that $\boldsymbol{r}$ is a **recursive presentation** of X if the relations on ω^3

$$P(i,j,k) \Leftrightarrow \mathrm{d}(r_i, r_j) \leqslant q_k \qquad Q(i,j,k) \Leftrightarrow \mathrm{d}(r_i, r_j) < q_k$$

are computable. The structure $\mathcal{X} = (X, \mathrm{d}, \boldsymbol{r})$ is called **recursively presented metric space**. If moreover (X, d) is complete, then $\mathcal{X}$ is called **recursively presented Polish space**.

For any recursively presented metric space we fix the following enumeration of the basis:

$$V_i^{\mathcal{X}} = B_{q_{(i)_1}}(\boldsymbol{r}_{(i)_0}) = \{x \in X \mid d(x, \boldsymbol{r}_{(i)_0}) < q_{(i)_1}\}$$

where $(q_l)_{l\in\omega}$ is a fixed computable enumeration of $\mathbb{Q}^+$.

Remark

Having fixed a basis, we can associate to $\mathcal{X}$ the canonical neighborhood representation $\rho_{\mathcal{X}} : \omega^\omega \rightharpoonup \mathcal{X}$ *defined by:*

$$\rho_{\mathcal{X}}(p) = x \Leftrightarrow \mathrm{ran}(p) = \mathrm{N}_{\mathrm{base}}(x) \coloneqq \{n \in \omega \mid x \in V_n^{\mathcal{X}}\}$$

Σ^0_1 sets and Σ^0_1-recursive functions

Definition

A subset A of a r.p.m.s. $\mathcal{X}$ is called $\Sigma^0_1(\mathcal{X})$ (also said **effectively open** in $\mathcal{X}$) if there is a c.e. set A^* in ω such that

$$A = \bigcup_{n \in A^*} V_n^{\mathcal{X}}$$

The Σ^0_1 sets of the r.p.m.s. $(\omega, (\{n\})_{n\in\omega})$ are exactly the c.e. sets.

Definition

Given $\mathcal{X}$, $\mathcal{Y}$ r.p.m.s., a function $f : \mathcal{X} \to \mathcal{Y}$ is **Σ^0_1-recursive** if its diagram is Σ^0_1, that is:

$$D_f = \{(x, n) \mid f(x) \in V_n^{\mathcal{Y}}\} \in \Sigma^0_1(\mathcal{X} \times \omega)$$

Remark

- *Any Σ^0_1-recursive function is also continuous.*
- *Σ^0_1-recursive functions generalize not only the computable functions on the natural numbers but also the computable functions on ω^ω (Type-2 theory of effectivity).*
- *The Σ^0_1-recursive functions coincide with the $(\rho_{\mathcal{X}}, \rho_{\mathcal{Y}})$-computable functions induced by the canonical neighborhood representation.*

Continuous degrees

Definition

Given $\mathcal{X}$ and $\mathcal{Y}$ r.p.m.s., $y \in Y$ is **representation reducible** to $x \in X$ (and we write $y \leqslant_M x$) if there is some $e \in \omega$ such that $\Phi_e^{\mathcal{X},\mathcal{Y}}(x) = y$, where $\Phi_e^{\mathcal{X},\mathcal{Y}}$ is the e-th partial Σ_1^0-recursive function on its domain.

$\leqslant_M$ is a quasi-order (reflexive and transitive).

Definition (Continuous degrees [Mil04] and [GKN21])

Given $\mathcal{X}$ r.p.m.s., the **continuous degree** of $x \in \mathcal{X}$ is its equivalence class under the relation $\equiv_M$ (over elements of recursive spaces).

Fact

Given $\mathcal{X}$, $\mathcal{Y}$ r.p.m.s.,

1. $\forall\, x, y \in \omega^\omega (y \leqslant_T x \Leftrightarrow y \leqslant_M x)$
2. $\forall\, x \in X\ \forall\, z \in \omega^\omega (x \leqslant_M z \Leftrightarrow\ \exists\, p \in \omega^\omega (\rho_{\mathcal{X}}(p) = x \wedge p \leqslant_T z))$

Universal sets

Definition

Given $\mathcal{X}$ and $\mathcal{Y}$ r.p.m.s., $G \in \Sigma_1^0(\mathcal{X} \times \mathcal{Y})$ is **universal** for $\Sigma_1^0(\mathcal{Y})$ if

$$\forall\, P \subseteq Y(P \in \Sigma_1^0 \Leftrightarrow \exists\, x \in X(P = G_x))$$

where $G_x = \{y \in Y \mid G(x, y)\}$ is called x-section of G.

Definition

A **universal system** for Σ_1^0 sets is a class-function $\mathcal{Y} \mapsto G^{\mathcal{Y}} \subseteq \omega \times Y$ such that $G^{\mathcal{Y}}$ is universal for every r.p.m.s. $\mathcal{Y}$.

A particular universal system

Denote with $W_e = \{n \in \omega \mid \varphi_e(n) \downarrow\}$ the e-th c.e. set of ω. Given a r.p.m.s. space $\mathcal{Y}$, we define:

$$H^{\mathcal{Y}}_{\Sigma^0_1} = \{(e, y) \in \omega \times Y \mid \exists\, n \in W_e(y \in V^{\mathcal{Y}}_n)\}$$

Each $H^{\mathcal{Y}}_{\Sigma^0_1}$ is universal for $\Sigma^0_1(\mathcal{Y})$, hence $(H^{\mathcal{Y}}_{\Sigma^0_1})_{\mathcal{Y}}$ is an *universal system*.
The reason why we are interested in this specific universal system is because it has the following property, which is basically the s-m-n theorem:

Definition

A universal system $(G^{\mathcal{Y}})_{\mathcal{Y}}$ is **effectively good** if for every $k \in \omega$, and every r.p.m.s. $\mathcal{Y}$, there exists a computable function $S : \omega^{k+1} \to \omega$ such that:

$$\forall\, e \in \omega\, \forall\, x \in \omega^k\, \forall\, y \in Y(G^{\omega^k \times \mathcal{Y}}(e, x, y) \Leftrightarrow G^{\mathcal{Y}}(S(e, x), y))$$

As any separable metrizable has a structure of ε-r.p.m.s. (for a strong enough oracle $\varepsilon \in \omega^\omega$), we can extend this construction to any separable metrizable space.

The Turing jump on Polish spaces

The notion of Turing Jump can be extended to recursively presented metric spaces:

Definition ([GKN21])

Given $\mathcal{X}$ r.p.m.s., the $\mathbf{\Sigma^0_1}$**-jump** is the function defined as:

$$\begin{aligned} J^{(1)}_{\mathcal{X}} &: \mathcal{X} \to 2^\omega \\ &x \mapsto J^{(1)}_{\mathcal{X}}(x) = \{e \in \omega \mid x \in H^{\mathcal{X}}_{\Sigma^0_1, e}\} \end{aligned}$$

Similarly, we can define the Σ^0_1-jump for ε-r.p.m.s..
The usual Turing Jump on the Baire space is "equivalent" to the Σ^0_1-jump $J^{(1)}_{\omega^\omega}$. In fact:

$$\forall\, x \in \omega^\omega (J^{(1)}_{\omega^\omega}(x) \equiv_T J(x))$$

For this reason, we simply write $x^{(n)}$ instead of $J_{n-1} \circ J^{(1)}_{\mathcal{X}}(x)$. Moreover, it satisfies:

Theorem (Generalized Posner-Robinson for recursively presented spaces)

Given $\mathcal{X}$ and $\mathcal{Y}$ r.p.m.s., for every $n \in \omega$

$$\forall\, x \in X\, \forall\, y \in Y \left(y \leqslant_M x^{(n)} \,\dot{\vee}\, \exists\, g \in 2^\omega (x \oplus y \oplus g \geqslant_M (g \oplus x)^{(n+1)}) \right).$$

A modification of Lutz's game

Given two functions $f : \omega^\omega \to \omega^\omega$ and $g : \mathcal{X} \to \mathcal{Y}$ (where $\mathcal{X}$ and $\mathcal{Y}$ are r.p.m.s.) we define the game $G_M(f, g)$ as follows: Player 2 first plays a code $e \in \omega$ corresponding to a Σ^0_1-recursive function. For the rest of the game, Player 1 plays each turn one digit of a real $x = x_0 x_1 \ldots \in \omega^\omega$ and Player 2 plays one digit of each of two reals, $b = b_0 b_1 \ldots \in \omega^\omega$ and $z = z_0 z_1 \ldots \in \omega^\omega$.

Player 1		x_0	x_1	...
Player 2	e	$b_0,\ z_0$	$b_1,\ z_1$	...

Player 2 wins if and only if b is a $\rho_\mathcal{X}$-name for an element $y \in \mathcal{X}$ (i.e. $b \in \mathrm{dom}(\rho_\mathcal{X})$) and

$$f(x) = \Phi_e^{(\mathcal{Y} \times \omega^\omega \times \omega^\omega), \omega^\omega}(g(\rho_\mathcal{X}(b)), x, z).$$

Remark

If f and g are Borel functions and the domain of the representation $\rho_\mathcal{X} : \omega^\omega \rightharpoonup \mathcal{X}$ is Borel, then the payoff of the game $G_M(f, g)$ is Borel.

Lemma (Lutz, Carroy, Nicolosi)

- *If Player 2 has a winning strategy for* $G_M(f,g)$ *then* $f \leqslant_w g$*.*
- *If Player 1 has a winning strategy for* $G_M(J_n, g)$ *then* $g \in \mathrm{dec}(\boldsymbol{\Sigma}^0_n)$*.*

Sketch of the proof.

- Set $\varphi(x) = \rho_{\mathcal{X}} \circ b(x)$ and set $\psi(w,x) = \Phi_e^{(\mathcal{Y}\times\omega^\omega\times\omega^\omega),\omega^\omega}(w,x,z(x))$ (where $b(x)$ and $z(x)$ are the continuous functions determined by the winning strategy of Player 2).
- The result follows from the following characterization:

$$\exists\, w \in \omega^\omega \;\forall\, x \in X (g(x) \leqslant_M (x \oplus w)^{(n-1)}) \Leftrightarrow g \in \mathrm{dec}(\boldsymbol{\Sigma}^0_n)$$

Indeed, disjointifying the covering given by:

$$B_m = \{x \in X \mid g(x) = \Phi_m^{(\mathcal{X}\times\omega^\omega),\mathcal{Y}}((x \oplus w)^{(n-1)})\}$$

one gets a partition witnessing $g \in \mathrm{dec}(\boldsymbol{\Sigma}^0_n)$.
Given τ the winning strategy for Player 1, we prove that $w = \tau$ witnesses the inequality which gives us the decomposition. Suppose, toward a contradiction, that:

$$\exists\, x_0 \in X \left(g(x_0) \nleqslant_M (x_0 \oplus \tau)^{(n-1)}\right).$$

Sketch of the proof. (continued).

WLOG we can find a $w \in \omega^\omega$ and a name b of x_0 such that

$$b \leqslant_T w \ \wedge \ \tau \leqslant_T w \ \wedge \ g(x_0) \nleqslant_M w^{(n-1)}.$$

Using the Generalized Posner-Robinson Theorem, we can find some $v \in 2^\omega$ such that

$$g(x_0) \oplus w \oplus v \geqslant_M (w \oplus v)^{(n)}.$$

Then Player 2 can win against τ, playing:

Player 1	$x = \tau * (e, b, w \oplus v)$	
Player 2	e	b, $w \oplus v$

Indeed, $w \oplus v$ can compute Player 1's moves as it computes both b and the strategy τ, then:

$$g(x_0) \oplus w \oplus v \geqslant_M (w \oplus v)^{(n)} \geqslant_M x^{(n)}$$

and this computation depends uniformly on the first element e played by Player 2, that is:

$$\forall\, e \in \omega \left(\Phi_a^{\mathcal{Y} \times \omega^\omega \times \omega, \omega^\omega} (g \circ \rho_{\mathcal{X}}(b), w \oplus v, e) = J_n(x) \right)$$

Therefore, using Kleene's Recursion Theorem for r.p.m.s., one gets the right e to play.
Hence, the strategy τ is not winning! $\lightning$ $\square$

When is the game determined?

Using Borel determinacy we get

Theorem (Lutz, Carroy, Nicolosi)

For every $n \geqslant 1$, given $\mathcal{X}$, $\mathcal{Y}$ r.p.m.s. such that $\rho_{\mathcal{X}}$ has Borel domain and $g : \mathcal{X} \to \mathcal{Y}$ Borel function; then either $g \in \mathrm{dec}(\boldsymbol{\Sigma}^0_n)$ or $J_n \leqslant_w g$.

Clearly, under the Axiom of Determinacy, this result generalizes to all r.p.m.s. and functions. In addition:

Fact

For a r.p.m.s., the domain of the canonical neighborhood representation $\rho_{\mathcal{X}}$ is Δ^1_1 if and only if X is Δ^1_1 when viewed as subset of its metric completion.

In topological terms, the domain of $\rho_{\mathcal{X}}$ is Borel exactly when X is *standard Borel*. Thus we get:

Corollary

For every $n \geqslant 1$, given X standard Borel, Y separable metrizable and $g : X \to Y$ Borel function; then either $g \in \mathrm{dec}(\boldsymbol{\Sigma}^0_n)$ or $J_n \leqslant_w g$.

What about Borel functions on analytic domains?

Recall that an *analytic space* is a separable metrizable space homeomorphic to an analytic subset of a Polish space. Similarly to the Borel case we have that:

Fact

For a separable metrizable space, the domain of the canonical neighborhood representation $\rho_{\mathcal{X}}$ is $\boldsymbol{\Sigma}^1_1$ if and only if X is an analytic space.

Therefore, Borel determinacy no longer directly applies when g is a Borel function defined on an analytic domain. A standard route to overcome this problem is to *unfold the game*, that is:

- Given $g : X \to Y$ with X analytic, consider the closed set $F \subseteq \omega^\omega \times \omega^\omega$ such that

$$b \in \mathrm{dom}(\rho_{\mathcal{X}}) \Leftrightarrow \exists\, q \in \omega^\omega((b, q) \in F)$$

- Ask Player 2 to produce, in addition to the b such that $b \in \mathrm{dom}(\rho_{\mathcal{X}})$, also a corresponding q witnessing $(b, q) \in F$.

However, there is no obvious way to choose such a witness q in a way that the z obtained by the Generalized Posner-Robinson computes it.

How to overcome this?

How to overcome this?

The following theorem is a relativization of [Ler17, Exercise 4.18].

Theorem ([HJS24, Corollary 3.17])

Let $x, y, q \in \omega^\omega$. If $y \nleqslant_T x$, then there is a $z \in \omega^\omega$ such that

$$x \leqslant_T z \ \wedge \ q \leqslant_T z' \ \wedge \ y \nleqslant_T z.$$

With some work, one can extend this result to:

Theorem (Jump coding cone avoidance)

Let $n \geqslant 1$, $x, q \in \omega^\omega$, and $y \in \mathcal{Y}$ be a point of a r.p.m.s.. If $y \nleqslant_M x^{(n-1)}$, then there exists $r \in \omega^\omega$ such that

$$x \leqslant_T r \ \wedge \ q \leqslant_T r^{(n)} \ \wedge \ y \nleqslant_M r^{(n-1)}.$$

The modified game

Given a Borel function $g : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X}$ and $\mathcal{Y}$ are r.p.m.s. and $\mathrm{dom}(\rho_{\mathcal{X}}) = \mathrm{proj}_1(F)$ for some closed set $F \subseteq \omega^\omega \times \omega^\omega$ define the game $G^{\mathrm{an}}_M(F, g)$ as follows: as before, Player 2 first plays a code $e \in \omega$ and, for the rest of the game, Player 1 plays each turn one digit of a real $x \in \omega^\omega$, but now Player 2 plays one digit of each of three reals: $b \in \omega^\omega$, $r \in \omega^\omega$ and $z \in \omega^\omega$.

Player 1		x_0	x_1	...
Player 2	e	$b_0, r_0,\ z_0$	$b_1, r_1,\ z_1$	...

Player 2 wins if and only if

- $\exists\ a \in \omega$ such that $(b, \Phi_a(r^{(n)})) \in F$ and
- $J_n(x) = \Phi_e^{(\mathcal{Y} \times \omega^\omega \times \omega^\omega), \omega^\omega}(g(\rho_{\mathcal{X}}(b)), x, z)$.

Remark

Under these assumptions, the payoff of the game $G^{\mathrm{an}}_M(F, g)$ is Borel.

Lemma

- *If Player 2 has a winning strategy for $G_M^{an}(F, g)$ then $J_n \leqslant_w g$.*
- *If Player 1 has a winning strategy for $G_M^{an}(F, g)$ then $g \in \mathrm{dec}(\mathbf{\Sigma}^0_n)$.*

Sketch of the proof.

- Since this game is harder for Player 2, the proof for the game $G_M(J_n, g)$ still works.
- Also in this case the proof is similar to the previous game. Suppose towards contradiction that:

$$\exists\, x_0 \in X \left(g(x_0) \nleqslant_M (x_0 \oplus \tau)^{(n-1)} \right).$$

as before we can find a $w \in \omega^\omega$ and a name b of x_0 such that

$$b \leqslant_T w \ \wedge\ \tau \leqslant_T w \ \wedge\ g(x_0) \nleqslant_M w^{(n-1)}.$$

Now choose $q \in \omega^\omega$ with $(b, q) \in F$. Applying the jump coding cone avoidance Theorem to $w, q, y = g(x_0)$, we obtain $r \in \omega^\omega$ such that

$$w \leqslant_T r \ \wedge\ q \leqslant_T r^{(n)} \ \wedge\ g(x_0) \nleqslant_M r^{(n-1)}.$$

So, after using the Generalized Posner-Robinson Theorem to the last inequality, we can conclude as in the original game. □

A counterexample: the analyticity of the domain is sharp!

Theorem (Guaspari, Kechris, Sacks)

Assume $V = L$. The set $\mathcal{C}_1 = \{x \in \omega^\omega \mid x \in L_{\omega_1^{\mathrm{CK},x}}\}$ is a Π^1_1 set which is Turing cofinal and thin *(i.e., it contains no nonempty perfect sets).*

Theorem

Assume $V = L$. For every $n \geqslant 1$, the Borel function $g = J_n \restriction \mathcal{C}_1$ satisfies

$$g \notin \mathrm{dec}(\boldsymbol{\Sigma}^0_n) \wedge J_n \nleqslant_w g.$$

Further directions

- Is there a counterexample to the metrizability assumption of the codomain?
- Can this result be strengthened up to the Generalized Solecki Dichotomy proved by Marks and Montalbán? What about the topological embedding?

Thank you for your attention.

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